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Rigid Interfaces for Lattice Models at Low Temperatures

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Lattice models (on a hypercubic lattice of dimension larger than or equal to three) with spins attaining a finite number of values and finite-range interactions at low temperatures are considered. The existence of rigid interfaces as well as of surface tension under appropriate conditions is proven and the properties of corresponding Gibbs states are investigated.

KEY WORDS: Lattice models at low temperatures; rigid interfaces; translation-noninvariant Gibbs states; surface tension; Pirogov–Sinai theory.

1. INTRODUCTION

A theory describing phase diagrams and translation-invariant Gibbs states at low temperatures is now well developed for lattice models with finite-range interaction exhibiting only a finite number of ground states. Starting from the early papers of Peierls, Griffith, Dobrushin, and others, it found one of its most general expressions in the Pirogov–Sinai theory.[(1,2)]

It is natural to ask about the existence (and description) of translation-noninvariant Gibbs states in comparably general situations. The first rigorous result in that direction was Dobrushin's paper[(3)] concerning an interface between phases of opposite magnetization in the Ising model. His approach was further pursued and applied in a series of papers.[(4),4] Dobrushin's strategy is to describe interfaces enforced by a suitable boun-

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[4] For an implementation of this method in lattice gauge theory see Ref. 5.

dary condition in terms of perturbations (called walls) of the flat interface. The corresponding Gibbs probability of interfaces thus forms in fact a certain $(\nu - 1)$-dimensional model ($\nu \geq 3$ is the dimension of the original lattice).

The aim of the present paper is to show that, when combined with the results and methods of Pirogov and Sinai,[1] Dobrushin's approach gives a reasonably general tool for the study of translation-noninvariant Gibbs states. The Pirogov–Sinai theory was used in a similar situation in a study of surface tension for two-dimensional models.[6] Here we need more detailed information, since we have to control the thermodynamic limit of the Gibbs state as well as its correlations.

When expressing the probability of an interface in terms of its energy, one has to take into account the corrections describing the influence of the surrounding pure phases. However, expanding the partition functions "above" and "below" an interface by means of corresponding contour models (Pirogov–Sinai theory), one gets an expression with corrections localized in the neighborhood of the interface. Thus, one finally gets a $(\nu - 1)$-dimensional model with "polymers" which are aggregates of walls and above-mentioned corrections. The study of probability of interfaces then turns into the study of a $(\nu - 1)$-dimensional polymer model of "usual type" (polymers with only a hard-core interaction), which may be studied with the help of a cluster expansion.[5]

Let us notice that the present paper is devoted to the study of models with a unique type of "flat interface" (up to translations). However, the method may be extended without much change to cover more general models exhibiting the phenomenon of "phase transition inside the interface." This is briefly outlined in Ref. 8. The only novel feature is that even the resulting $(\nu - 1)$-dimensional model is studied with the help of Pirogov–Sinai theory.

The paper is organized as follows.

In Section 2 we introduce some basic concepts and adjust the geometrical notions from Ref. 3 to our more general situation. We formulate our main results about the existence of a translation-noninvariant Gibbs state in Theorem 1. Its properties are described in a more detailed fashion in Theorem 2. Theorem 3 concerns the surface tension.

Section 3 is a brief review of the Pirogov–Sinai theory with some useful complements. The notation and abstract machinery of contour (polymer) models used here and throughout the paper are briefly recapitulated in Appendix B.

Section 4 is devoted to the transcription of the partition function to a

[5] A similar approach was recently used in other contexts, e.g., in Ref. 7.

form suitable for the expression of probabilities of interfaces in terms of "aggregate models" as indicated in the above explanation of our strategy.

In Section 5 we establish a condition of the type (B4) from Appendix B for aggregates; it enables us to apply the abstract theory of contour models for the aggregate model.

Translating the information about probabilities of aggregates into the language of walls (and interfaces) in Section 6, we prove Theorem 2. Theorem 1 follows from it with the help of some known facts about Gibbs states.

Section 7 contains a proof of existence of surface tension (Theorem 3) as well as its explicit expression at low temperatures in Proposition 7.1.

For the reader's convenience we include two Appendices. Appendix A contains a formulation and proofs of some "obvious" geometrical statements used throughout the paper. Appendix B (based essentially on Ref. 9) summarizes the results of the theory of contour models and corresponding cluster expansions.

2. SETTING AND RESULTS

2.1. Gibbs States

We consider lattice models on a ν-dimensional hypercubic lattice $\mathbb{Z}^\nu$ with ν an integer larger than 2. (Generalization to other lattices in $\mathbb{R}^\nu$ is possible). We always use the L_∞-metric on $\mathbb{Z}^\nu$: $\rho(i, j) = \max |i_k - j_k|$. In particular, a set $\Lambda \subset \mathbb{Z}^\nu$ is *R-connected* if any two points $i, j \in \Lambda$ are connected by a path $\{i^{(1)},..., i^{(k)}\} \subset \Lambda$, $i^{(1)} = i$, $i^{(k)} = j$, such that $\rho(i^{(l)}, i^{(l-1)}) \leqslant R$ for every $l = 2,..., k$. The *R-components* of a set in $\mathbb{Z}^\nu$ are its maximal R-connected subsets. If $R = 1$, we omit it from the notation (connected, component,...). Whenever $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^\nu$, their *distance* is

$$d(\Lambda_1, \Lambda_2) = \inf\{\rho(i, j) \mid i \in \Lambda_1, j \in \Lambda_2\}$$

(if $\Lambda_1 = \emptyset$, we define $d(\emptyset, \Lambda_2) = \infty$ for each $\Lambda_2 \subset \mathbb{Z}^\nu$). We say that $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^\nu$ are *R-distant* if $d(\Lambda_1, \Lambda_2) > R$ and denote $\Lambda_1 \subset\subset \Lambda_2$ whenever Λ_1 and $\Lambda_2^c = \mathbb{Z}^\nu \backslash \Lambda_2$ are distant. Finally, the *diameter* of $\Lambda \subset \mathbb{Z}^\nu$ is $\operatorname{diam} \Lambda = \sup\{\rho(i, j) \mid i, j \in \Lambda\}$.

Considered lattice models will always have a finite set S of spin values attached to each lattice site $i \in \mathbb{Z}^\nu$. Whenever $V \subset \mathbb{Z}^\nu$, we denote $X_V = S^V$ and, in particular, $X = S^{\mathbb{Z}^\nu}$. Let us make a convention that by $\Lambda \subset \mathbb{Z}^\nu$ we shall always denote nonempty finite sets. A lattice model with a formal Hamiltonian $H(x) = \sum \varphi_\Lambda(x)$ is defined by introducing an *interaction* $\varphi_\Lambda : X_\Lambda \to \mathbb{R}$ for each $\Lambda \subset \mathbb{Z}^\nu$. Denoting by x_V the restriction of $x \in X$ to

X_V, we shall often write $\varphi_A(x)$ instead of $\varphi_A(x_A)$. We suppose that the interaction is of *finite range* R, i.e., $\varphi_A = 0$ whenever diam $A > R$ for some positive integer R.

If $V \subset \mathbb{Z}^\nu$ is finite, $V \neq \emptyset$, $z \in X$, and $V^c = \mathbb{Z}^\nu \backslash V$, we introduce the "*physical*" *partition function*

$$Z(V \mid z; \beta H) = \sum_{x_V \in X_V} \exp\left[-\beta \sum_{A \cap V \neq \emptyset} \varphi_A(x_V \times z_{V^c}) \right]$$

Whenever x, $z \in X$ we shall denote the sum $\sum \varphi_A(x_V \times z_{V^c})$ either by $H_V(x_V \mid z_{V^c})$ or by $H_V(x \mid z)$ and call it the *Hamiltonian in the volume* V of the configuration x (x_V) under the boundary condition z (z_{V^c}).

Whenever $x \in X$ and $V \subset \mathbb{Z}^\nu$ is finite, we also introduce the *energy* in V of the configuration x by

$$E_V(x) = \sum \varphi_A(x) \frac{|A \cap V|}{|A|}$$

where $|A|$ denotes the number of lattice sites in A. The usefulness of this notion stems from its additivity: $E_{V_1 \cup V_2}(x) = E_{V_1}(x) + E_{V_2}(x)$ whenever $V_1 \cap V_2 = \emptyset$. In particular, we write $e_i(x) = E_{\{i\}}(x)$ for $i \in \mathbb{Z}^\nu$ and note that if x is a translation-invariant configuration, then $e(x) = e_i(x)$ is its specific energy.

If $V \subset \mathbb{Z}^\nu$ is finite and nonempty, we define the kernel $\mu_V^{\beta H}(x \mid z)$, called a Gibbs state in V, under the boundary condition z_{V^c} by the formula

$$\int f(x)\, \mu_V^{\beta H}(dx \mid z) = \sum_{x_V} f(x_V \times z_{V^c}) \frac{\exp[-\beta H_V(x_V \mid z)]}{Z(V \mid z; \beta H)}$$

for every measurable bounded function f. Instead of $\mu_V^{\beta H}(x \mid z)$ we sometimes use the notation $\mu_V(x \mid z)$ or $\mu_V(x \mid z_{V^c})$.

Let now $V \subset \mathbb{Z}^\nu$ be possibly infinite. We say that a probability measure μ on X (equipped with the σ-algebra generated by cylinder sets) is a *Gibbs state in the volume* V of a Hamiltonian H and at an inverse temperature β if

$$\mu(f) = \int \left[\int f(x)\, \mu_A^{\beta H}(dx \mid y) \right] \mu(dy)$$

whenever A finite $\subset V$ and f is a measurable bounded function. The above definition applies in particular to $V = \mathbb{Z}^\nu$. A Gibbs state μ in V is a Gibbs state in V with a boundary condition $z_{V^c} \in X_{V^c}$ if

$$\mu(\{x \in X \mid x_{V^c} = z_{V^c}\}) = 1$$

Note that if $V \neq \emptyset$ is finite, there is a unique Gibbs state in V under a boundary z_{V^c}, namely the state μ for which $\mu(f) = \int f(x)\, \mu_V^{\beta H}(dx \mid z)$. This justifies the term we used for the kernel $\mu_V^{\beta H}$. Later we use also the following:

Lemma 2.1. *If* $V \subset \mathbb{Z}^\nu$ is nonempty and either (i) V has only finite R-components, or (ii) V is a cylinder with a finite base, i.e., there exists a finite $B \subset \mathbb{Z}^{\nu-1}$ such that $V = \{i \in \mathbb{Z}^\nu \mid i_1 \in \mathbb{Z},\ (i_2,..., i_\nu) \in B\}$, then there exists a unique Gibbs state $\mu_V^{\beta H}(dx \mid z_{V^c})$ in V under a boundary condition $z_{V^c} \in X_{V^c}$.

A proof is straightforward for (i) and is given, e.g., in Ref. 10 for (ii).

2.2. Interface

In the spirit of Pigorov–Sinai theory, the "excitations" of a configuration will be considered by comparing it with a chosen set of translation-invariant configurations $\{x^1,..., x^r\}$, $r \geqslant 2$. A similar role for an investigation of an interface is played by a fixed configuration y^{12} fulfilling the following conditions:

(i) y^{12} is horizontally translation-invariant: $y^{12}(i) = y^{12}(j)$ whenever $i_1 = j_1$.

(ii) $y^{12}(i) = x^1(i)$ whenever i_1 is large enough and $y^{12}(i) = x^2(i)$ whenever $-i_1$ is large enough.

We shall often denote y^{12} simply by y and we always reserve this letter for this particular configuration.

Our goal is to investigate a Gibbs state, which could be constructed as a weak limit of Gibbs states in finite volumes V under the boundary condition $y^{12}_{V^c}$. In particular, we look for conditions on interactions $\{\varphi_A\}$ and (inverse) temperature β that would ensure that the considered Gibbs state is not translation-invariant. In a manner similar to Dobrushin,[3] whose method we folow, we shall prove more about the structure of this Gibbs state. To describe it we use Pirogov–Sinai contours and generalize several "geometrical" notions from Ref. 3 to our situation.

Definition 2.1. Let x be a configuration, $x \in X$.

(i) A hypercube $C \subset \mathbb{Z}^\nu$ of diameter R is called *good* (for the configuration x) if $x_C = x^q_C$ for some $q = 1,..., r$. Otherwise, it is a *bad* hypercube. The *boundary* $B(x)$ is the union of all bad hypercubes of x.

(ii) If Γ is a finite component of $B(x)$, we call the pair $\gamma = (\Gamma, x_\Gamma)$ a *contour of* x and $\Gamma = \operatorname{supp} \gamma$ its support. The pair $\gamma = (\Gamma, x_\Gamma)$ is called a *contour* if it is a contour of some $x \in X$. Whenever γ is a contour, we always denote its support by Γ. Denote also $\text{Ext}\,\gamma$ and $\text{Int}\,\gamma$ the (only) infinite component of $\mathbb{Z}^\nu \backslash \Gamma$ and the union of its finite components, respectively.

Let us note that $E_\Gamma(x)$ does not depend on the choice of $x \in X$ with a contour γ, and we may and shall denote it by $E_\Gamma(\gamma)$. Let us recall also that there exists a unique $q \in \{1,..., r\}$ such that there exists a configuration with $x_{\text{Ext}\,\gamma} = (x^q)_{\text{Ext}\,\gamma}$ which has γ as its contour. This allows us to call γ a q-contour. We shall use the subscript q to indicate this fact, denoting thus, e.g., the set of all q-contours by $\mathbb{K}_q$ (cf. Appendix B).

In the following we suppose that the boundary $B(y^{12})$ has a unique infinite component to be denoted by $I(y)$ or sometimes I_0. Later (in the next subsection), we show that this assumption actually follows from our "wall Peierls condition." An interface can be defined for a class of configurations:

Definition 2.2. Let x be such that $B(x)$ has a unique infinite component $I(x)$, $\mathbb{Z}^\nu \backslash I(x)$ has exactly two infinite R-components, and $I(x) \backslash I(y)$ has only finite R-components. Then $\mathbb{I}(x) = (I(x), x_{I(x)})$ will be called a (y-) *interface of* x. A pair $\mathbb{I} = (I, x_I)$ with $I \subset \mathbb{Z}^\nu$ and $x_I \in X_I$ is a (y-) *interface* if there exists $x \in X$ such that $\mathbb{I} = \mathbb{I}(x)$. We always use I for support of $\mathbb{I}$, i.e., $\mathbb{I} = (I, x_I)$, and denote by $\mathbb{Z}_1^\nu(I)$ [resp. $\mathbb{Z}_2^\nu(I)$] the "upper" (resp. "lower") infinite R-component of $\mathbb{Z}^\nu \backslash I$ and by $\text{Int}_\alpha I$ its remaining (finite) R-components. Whenever $V \subset \mathbb{Z}^\nu$ we denote by V_R the neighborhood $V_R = \{i \mid d(i, V) < R + 1]$ and by $V_q(I)$, $q = 1, 2$, the intersection $V_q(I) = \mathbb{Z}_q^\nu(I) \cap V_R$.

We prove in Appendix A that configurations differing from y only on a finite set have an interface (cf. Lema 4.1).

Let T_h denote the vertical shift by $h \in \mathbb{Z}$, i.e., $T_h(i) = (i_1 + h, i_2,..., i_\nu)$ for $i \in \mathbb{Z}^\nu$ and $T_h(x)(i) = x(T_h^{-1}(i))$ for $x \in X$, $T_h(M, x_M) = (T_h(M), T_h(x_M))$ for $M \subset \mathbb{Z}^\nu$, $x_M \in X_M$. Whenever $A \subset \mathbb{Z}^\nu$, we shall denote $\pi^{-1}(A) = \{j = T_h(i) \mid u \in A,\ h \in \mathbb{Z}\}$; $\pi(A) = \pi^{-1}(A) \cap T_0$. We shall now introduce ceilings and walls of an interface $\mathbb{I}$.

Definition 2.3. Let $\mathbb{I} = (I, x)$ be an interface. A set $C \subset I$ is a *column* (of $\mathbb{I}$) if $\pi^{-1}(\{i\}) \cap I = C$ whenever $i \in C$ and there exists $h(C) \in \mathbb{Z}$ such that $C = \pi^{-1}(C) \cap T_{h(C)}(I_0)$ and $x_C = (T_{h(C)}(y))_C$. The number $h(C)$ is called the height of C. A *ceiling column* is a column C such that $C_R \cap I \equiv \{i \in \mathbb{Z}^\nu \mid d(i, C) < R + 1\} \cap I$ is a union of columns. A *ceiling* C of $\mathbb{I}$ is a component of the union of all ceiling columns.

A pair $\mathbb{w} = (W, x_W)$ is a *wall* of $\mathbb{I}$ if $W = \text{supp}\,\mathbb{w}$ is a component of $I \backslash \bigcup (C \mid C$ is a ceiling of $\mathbb{I})$ and $x_W = (x_I)_W$. A pair $\mathbb{w} = (W, x_W)$ is called a wall if it is a wall of some interface. We denote the support of a wall $\mathbb{w}$ always by W.

Let $\mathbb{w}$ be a wall. Then, for any interface (and a configuration) $\mathbb{I}(x)$ such that $\mathbb{w}$ is its wall, the number $E_W(x)$ is the same. We denote it by $E_W(\mathbb{w})$.

2.3. A Gibbs State with an Interface

Let us collect here our main assumption about interactions $\{\varphi_A\}$ (as well as inverse temperature β):

(I) Interactions $\{\varphi_A\}$ are translation-invariant and of finite range R.

(CP) *Contour Peierls condition* (GPS condition): There exists $\rho_1 > 0$ such that

$$E_\Gamma(\gamma) - E_\Gamma(x^q) \geqslant \rho_1 |\Gamma|$$

whenever $\gamma = (\Gamma, x_\Gamma)$ is a q-contour.

(WP) *Wall Peierls condition*: There exists $\rho_2 > 0$ such that

$$E_W(\mathbb{w}) - [E_{\pi(W)}(y) + e(|W| - |\pi(W)|)] \geqslant \rho_2 |W|$$

with either $e = \min\{e(x^q)\}$, $e = e(x^1)$, or $e = e(x^2)$, whenever $\mathbb{w} = (W, x_W)$ is a wall.

(S) The configurations x^1 and x^2 correspond to stable phases for interactions $\{\varphi_A\}$ at an inverse temperature β.

We recall the Pirogov–Sinai theory, and the notion of stability in particular, in Section 3. Typically we have a situation familiar from the Pirogov–Sinai theory with a Hamiltonian H arising as a small perturbation of some H_0 complying with (I), (CP), and (WP), by an addition of certain "external fields." If this perturbation is small, the Peierls conditions (CP) and (WP) are again satisfied (perhaps with a slightly smaller ρ_1 and ρ_2). Whenever β is large enough, the "external fields" may be adjusted to satisfy (S).

Let us notice that the lhs in (WP) do not depend on other walls of the interface or on its vertical shift and thus one may supose, when verifying (WP), that $\mathbb{w}$ is an only wall of an interface. Notice also that the alternative inequalities in (WP) employing $e(x^1)$ or $e(x^2)$ instead of $\min_{q=1,\dots,r} e(x^q)$ may lead to a slightly larger ρ_2 and thus to stronger statements in a (not *a priori* excluded) case with $|W| < |\pi(W)|$.

An immediate consequence of the wall Peierls condition is the above mentioned fact that $B(y^{12})$ has only one infinite component. Indeed, if it were not the case and $B(y^{12})$ consisted of at least two disjoint horizontal layers, one could construct a new interface y' by shifting one of those strips vertically in such a way that, assuming for brevity that all $e(x^q)$ are the same,

$$|H_V(y' \mid y) - H_V(y \mid y)| \leqslant \text{const} \cdot L^{\nu-2}$$

for large hypercubes V with side L.

The interface corresponding to the configuration $y'_V \times y_{V^c}$ could have only one wall $\mathbb{w}$ for $E(\mathbb{w}) \sim L^{\nu-2}$, while the second term of the left-hand side in (WP) would be of order $L^{\nu-1}$, in clear contradiction with (WP).

In some models the stability (S) at large β is due to a symmetry between configurations x^1 and x^2. Included here in particular are models for which an interface has been studied before: the Ising model in Refs. 3, 4, and 11 and Widom–Rowlinson models in Ref. 4.

It is instructive to analyze the case of the Ising antiferromagnet. Although a theorem about the existence of an interface has been announced,[12] it seems, as we shall indicate below, that it cannot (at least for the case with a nonvanishing external field) be considered as a straightforward generalization of the approach used for the ferromagnet. Strictly speaking, the Ising antiferromagnet does not belong to the class of models considered here, since its ground configurations are not translation-invariant. However, this failing may be remedied by partitioning the lattice into blocks of 2^ν sites each, attaching a new spin variable attaining 2^{2^ν} values to every block, and rewriting the interactions in an obvious way in terms of new block spins. The ground configurations x^1, x^2 expressed in terms of new block spins are already translation-invariant. However, what fails is the wall Peierls condition. Namely, there are two natural and different generic interfaces: the first corresponds to the configuration y^{I}, for which, in terms of original spins, $y^{\mathrm{I}}(i) = x^1(i)$ whenever $i_1 > 1/2$ and $y^{\mathrm{I}}(i) = x^2(i)$ otherwise; the second corresponds to the configuration y^{II}, for which $y^{\mathrm{II}}(i) = x^1(i)$ whenever $i_1 > 3/2$ and $y^{\mathrm{II}}(i) = x^2(i)$ whenever $i_1 < 3/2$. When expressed in terms of new block spins, their difference is not simply due to a vertical translation (in the block lattice). Indeed, while in one of them, blocks with configuration x^1 on them touch directly blocks with configuration x^2, in the second one they are separated by a layer of blocks with configuration differing on them from both x^1 and x^2. Choosing one of the interfaces of y^{I} and y^{II} as our y^{12}, all areas of the second interface should be considered as walls from the point of view of our definition. On the other hand, it is clear that one pays for them by an energy proportional only to their fringe; hence, the wall Peierls condition cannot be satisfied. Thus, we see that a natural way to study the Ising antiferromagnet is to model ceilings upon both those interfaces. However, to do it one should generalize the method to the case of two (or generally a finite number of) different types of ceilings. The novel feature is that walls "remember" the type of ceiling outside of them and one has to take this fact into account when matching walls together. This is reminiscent of the situation with contours in the Pirogov–Sinai theory and actually it may be tackled in a similar way. While we are planning to study this case in a future publication, in the present article we confine ourselves to the case of one type of ceiling.

Note, finally, that an attempt to include the Ising antiferromagnet by generalizing to a theory with periodic configurations x^q, $q = 1,..., r$, would again lead to a theory with two types of ceilings. To see this, observe that since an odd translation transforms x^1 into x^2, a ceiling shifted together with the configuration vertically by an odd $h \in \mathbb{Z}$ will be a natural ceiling only after changing the configuration on it. In other words, from the configuration of a wall one can infer whether the wall matches outside ceilings only on odd (resp. even) levels. Thus, we have two types of ceilings according to the parity of their level.

Coming back to our situation, let us recall that if U is a cylinder with finite base, then there is a unique Gibbs state $\mu_U^{\beta H}(\cdot \mid y)$ in U under the boundary condition $y_{U^c} = y_{U^c}^{12}$ by Lema 2.1(ii). We shall consider its weak limit over the net of cylinders $U \subset \mathbb{Z}^\nu$ with finite bases ordered by inclusion (notation lim fin cyl). By the weak limit we mean a limit on all continuous functions on X, or, which is the same in our case, a limit on cylinder functions on X, i.e., such φ: $X \to \mathbb{R}$ that there exists a finite $\Lambda \subset \mathbb{Z}^\nu$ such that if $x, z \in X$, $x_\Lambda = z_\Lambda$, then $\varphi(x) = \varphi(z)$ (φ is living in Λ). By a constant $c = c(\nu, |S|, R)$ we shall always denote the constant from the estimate [Apendix B, formula (B.10)] on the number of contours of a given length

$$|\{\gamma \in |\mathbb{K}_q| \, \Gamma \ni i, \, |\Gamma| = n\}| \leqslant c^n$$

Let us introduce also a "thickness" of I_0 by $t = |\pi(\{i\})|/2R$ for any $i \in \mathbb{Z}^\nu$ (notice that $t \geqslant 1$).

Theorem 1. Let $\nu \geqslant 3$ and let $H \equiv \{\varphi_\Lambda\}$ satisfy the conditions (I), (CP), and (WP). There exist constants $c_1 = c_1(\nu, |S|, R, t)$ and $c_2 = c_2(\nu, |S|, R, t)$ such that if $\beta\rho_1 > c_1$, $\beta\rho_2 > c_2$, $V \subset \mathbb{Z}^\nu$ is a cylinder (not necessarily with a finite base), and $\{\varphi_\Lambda\}$ together with β satisfy (S), then

$$\mu = \text{lim fin cyl } \mu_U^{\beta H}(\cdot \, y)$$

exists.

Moreover, μ is a Gibbs state in V and (i) is extremal in the convex set of Gibbs states in V, (ii) is horizontally translation-invariant, (iii) is not translation-invariant, and (iv) μ-almost every configuration $x \in X$ has a y-interface $\mathbb{I}(x)$.

The above theorem is a direct consequence of Theorem 2, which includes a more detailed description of the state μ [it is natural to denote it again by $\mu_V^{\beta H}(\cdot \mid y)$]. Its proof is given in Section 6.

Our actual estimates (surely not optimal ones!) lead to the values

$$c_1 = 4 + \left(\frac{3}{2} + t\right)\log(2c) + \frac{\nu \log(2R+1)}{(2R+1)^\nu} + \left(\frac{3}{2}t + 1\right)\log 2 + t\,\frac{\log[3c(\nu-1)]}{2R} + t$$

$$c_2 = 3^\nu + \log(2c) + \max\left(5,\ 2t + \log 2 + \frac{\log[3c(\nu-1)]}{2R}\right)$$

with the constant $c(\nu-1)$ steming from an upper bound $[c(\nu-1)]^n$ on the number of connected subsets in $\mathbb{Z}^{\nu-1}$ of cardinality n containing a fixed site in $\mathbb{Z}^{\nu-1}$ [cf. the discussion of (B10) in Appendix B].

Let us remark that, as in Ref. 10, it can be shown that the state μ also can be gained by a limit over rectangular parallelepipeds with the ratio of their sides within certain bounds.

Referring to a situation typical for the application of the Pigorov–Sinai theory, which is explained in some detail in Section 3.1, we have the following straightforward Corollary of the above Theorem and Proposition 3.3.

Corollary. Let $\nu \geq 3$ and let H_0 satisfy the assumptions (I), (CP), and (WP) with respect to a collection of its translation-invariant ground configurations $\{x^1,..., x^r\}$ and let H_s, $s = 1,..., r-1$, satisfying (I) be such that $H_\mu = H_0 + \sum \mu_s H_s$ completely removes the degeneracy (see Section 3.1). Let us denote $K = \sup_{s = 1,...,r-1} \|H_s\|$ with

$$\|H\| = \sup_{i \in \mathbb{Z}^\nu} |E_{\{i\}}(x)|$$

and consider $\varepsilon < \min(\rho_1/2K,\ \rho_2/(1+2t)K)$, $\beta(\rho_1 - 2K\varepsilon) > c_1$, and $\beta(\rho_2 - (1+2t)K\varepsilon) > c_2$.

Then there is an $(r-2)$-dimensional surface S_β in the ball $U_0(\varepsilon) = \{\mu \in \mathbb{R}^{r-1} \mid \sum |\mu_s| < \varepsilon\}$ in the space of parameters, such that whenever $\mu \in S_\beta$, the Hamiltonian H_μ together with the (inverse) temperature β satisfies the conditions (I), (CP) with the constant $\rho_1 - 2K\varepsilon$, (WP) with the constant $\rho_2 - (1+2t)K\varepsilon$, and (S), and the statements of Theorems 1 and 2 hold.

2.4. Interface in Terms of an Admissible Family of Standard Walls

To state Theorem 2, we shall need some additional notions.

Definition 2.4. A pair $\mathbb{w} = (W, x_W)$ with $W \subset \mathbb{Z}^\nu$ and $x_W \in X_W$ is a standard wall if there exists an interface $\mathbb{I}_{\mathbb{w}}$ such that $\mathbb{w}$ is its only wall. We denote by $\mathbb{W}$ the set of all standard walls. A family $\mathbb{V}$ of walls is *compatible* if $\pi(\operatorname{supp} \mathbb{w}_1)$ and $\pi(\operatorname{supp} \mathbb{w}_2)$ are distant whenever $\mathbb{w}_1, \mathbb{w}_2 \in \mathbb{V}$. The set of all compatible families of standard walls will be denoted by $\mathscr{W}^{\mathrm{co}}$. If $\mathbb{w} = (W, x_W)$ is a wall, then $I_0 \backslash \pi(W)$ has one infinite component, to be denoted by $\operatorname{Ext}_{I_0}(\pi(W))$; $\operatorname{Int}_{I_0}(\pi(W)) = I_0 \backslash (\pi(W) \cup \operatorname{Ext}_{I_0} \pi(W))$. Let $\mathbb{w}_1$ and $\mathbb{w}_2$ be compatible walls, $W_k = \operatorname{supp} \mathbb{w}_k$, $k = 1, 2$. We say that $\mathbb{w}_1$ is *inside* of $\mathbb{w}_2$ if $\pi(W_1) \subset \operatorname{Int}_{I_0}(\pi(W_2))$. The set $\mathbb{E}(\mathbb{V})$ of *external walls of a compatible family* $\mathbb{V}$ of walls is a subset $\mathbb{E} \subset \mathbb{V}$ of those $\mathbb{w} \in \mathbb{V}$ for which $\mathbb{w}' \neq \mathbb{w}$ implies $\pi(W) \subset \operatorname{Ext}_{I_0}(\pi(W'))$. The set of all families $\mathbb{V} \in \mathscr{W}^{\mathrm{co}}$ such that $\mathbb{E}(\mathbb{V}) = \mathbb{V}$ will be denoted by $\mathscr{W}^{\mathrm{e}}$. A compatible family $\mathbb{V}$ of walls is admissible if every wall from $\mathbb{V} \backslash \mathbb{E}(\mathbb{V})$ is inside only a finite number of walls from $\mathbb{V}$. The set of all admissible families of standard walls will be denoted by $\mathscr{W}^{\mathrm{a}}$. If $\mathbb{V}$ is a compatible family of walls, we denote $\operatorname{supp} \mathbb{V} = \bigcup_{\mathbb{w} \in \mathbb{V}} \operatorname{supp} \mathbb{w}$ and $\|\mathbb{V}\| = |\operatorname{supp} \mathbb{V}|$.

We prove the following in Appendix A:

Lemma 2.2. (a) For every wall $\mathbb{w}$ there is one and only one $h = h(\mathbb{w}) \in \mathbb{Z}$ such that the shift $T_h \mathbb{w}$ is in $\mathbb{W}$. The shift $T_h \mathbb{w}$ is called $\mathbb{w}$ in the standard position.

(b) The mapping that ascribes to an interface $\mathbb{I}$ the collection of its walls in standard positions ($\mathbb{W}(\cdot)$) maps $\mathscr{I}$, the set of all interface, into $\mathscr{W}^{\mathrm{co}}$. It is one to one from $\mathscr{I}^{\mathrm{a}} = \mathbb{W}^{-1}(\mathscr{W}^{\mathrm{a}})$ (to be called the set of admissible interfaces) onto $\mathscr{W}^{\mathrm{a}}$.

Let us note that the existence of just one type of ceiling is crucial for this lemma.

2.5. A More Detailed Description of the Interface Gibbs State

Consider now the set $\mathscr{K}_q^{\mathrm{e}}(V)$ of all families θ of mutually external q-contours such that $\operatorname{supp} \theta \subset\subset V$, with $V \subset \mathbb{Z}^\nu$ not necessarily finite. Recall that two contours γ_1, γ_2 are external if $(\operatorname{Ext} \gamma_1)^{\mathrm{c}}$ and $(\operatorname{Ext} \gamma_2)^{\mathrm{c}}$ are distant. Let us observe that the set of all subsets of $\mathbb{K}_q$ may be identified with the compact metric space $\{0, 1\}^{\mathbb{K}_q}$. Endowing it with its Borel σ-algebra, the set $\mathscr{K}_q^{\mathrm{e}}(V)$ may be identified with its measurable subspace. Similarly, the sets $\mathscr{W}^{\mathrm{co}}$ and $\mathscr{W}^{\mathrm{a}} \approx \mathscr{I}^{\mathrm{a}}$ may be considered as measurable subspaces of the space of subsets of the set of all standard walls. We denote

$$\mathscr{I}(V) = \{\mathbb{I} \in \mathscr{I} \mid \mathbb{I} = \mathbb{I}(x_V \times y_{V^c}) \text{ for some } x_V \in X_V\}$$

whenever $V \subset \mathbb{Z}^\nu$. If V is a cylinder, i.e., $\pi^{-1}(V) = V$, then it is easy to see that

$$\mathscr{I}(V) = \left\{ \mathbb{I} \in \mathscr{I} \,\middle|\, \bigcup_{W \in \mathbb{W}(\mathbb{I})} W \subset V_R \right\}$$

Let $\mathbb{I} = (I, x_I)$ be an interface. In Definition 2.2 we introduced the R-components $V_1(I)$, $V_2(I)$, and $\mathrm{Int}_\alpha I$ of I^c. Let us consider $\theta_q \in \mathscr{K}_q^e(V_q(I))$, $q = 1, 2$, and denote

$$J = \left(\bigcup \mathrm{Int}_\alpha\, I \right) \cup \left(\bigcup_{\gamma \in \theta_1 \cup \theta_2} \mathrm{Int}\, \gamma \right)$$

and J^0 the maximal subset of J which is $(R+2)$-distant from J^c. By $X(\mathbb{I}, \theta_1, \theta_2)$ we denote the set of configurations $x \in X$ such that $\mathbb{I}(x) = \mathbb{I}$, and the set of external contours of x inside $V_q(I)$ is θ_q, $q = 1, 2$. Let us observe that $X(\mathbb{I}, \theta_1, \theta_2)$ is nonempty and whenever $x \in X(\mathbb{I}, \theta_1, \theta_2)$, its restriction $x_{(J^0)^c}$ to $\mathbb{Z}^\nu \setminus J^0$ is fixed. The set J^0 has only finite R-components and thus, according to Lema 2.1(i), there is a unique Gibbs state in J^0 under the boundary condition $x_{(J^0)^c}$. Let us denote it by $\mu(\cdot \mid \mathbb{I}, \theta_1, \theta_2)$ and note that $\mu(X(\mathbb{I}, \theta_1, \theta_2) \mid \mathbb{I}, \theta_1, \theta_2) = 1$. The following theorem is a refinement of Theorem 1.

Theorem 2. Let the assumptions of Theorem 1 or its Corollary be satisfied. Then the limit

$$\mu_V^{\beta H}(\cdot \mid y) = \lim \text{fin cyl}\, \mu_U^{\beta H}(\cdot \mid y)$$

exists and there exist probabilities $P_V^{\mathscr{I}}$ on $\mathscr{I}^a(V)$ for cylinders $V \subset \mathbb{Z}^\nu$ and $P_{q,V}^e$ on $\mathscr{K}_q^e(V)$, $q = 1, 2$, for arbitrary $V \subset \mathbb{Z}^\nu$, such that for each bounded, measurable function φ on X

$$\mu_V^{\beta H}(\varphi \mid y) = \int_{\mathscr{I}(V)} P_V^{\mathscr{I}}(d\mathbb{I}) \int_{\mathscr{K}_1^e(V_1(I)) \times \mathscr{K}_2^e(V_2(I))} \mu(\varphi \mid \mathbb{I}, \theta_1, \theta_2) \times [P_{1, V_1(I)}^e(d\theta_1) \otimes P_{2, V_2(I)}^e(d\theta_2)] \tag{2.1}$$

Moreover, if we take

$$\tau \leqslant \beta\rho_1 - 2$$

$$\omega \leqslant \beta\rho_1 - \left[3 + \log(2c) + 2\frac{\nu \log(2R+1)}{(2R+1)^\nu} \right]$$

$$\bar{\omega} \leqslant \min(\beta\rho_1 - c_1, \beta\rho_2 - c_2) + \frac{1}{2}\log(2c) + \frac{t}{2}\log 2$$

then: (i) Denoting $\rho_V^{\mathscr{I}}(\mathbb{V}) = P_V^{\mathscr{I}}(\{\mathbb{I} \in \mathscr{I}(V) \mid \mathbb{W}(\mathbb{I}) \supset \mathbb{V}\}$ whenever $\mathbb{V} \in \mathscr{W}^{\mathrm{co,f}}(V)$ and V is a cylinder, we have:

(a) $\rho_V^{\mathscr{I}}(\mathbb{V}) \leqslant \exp[-(2\bar{\omega} - t\log 2)\|\mathbb{V}\|]$

(b)
$$|\rho_{V_1}^{\mathscr{I}}(\mathbb{V}) - \rho_{V_2}^{\mathscr{I}}(\mathbb{V})| \leqslant 4\exp[-(\bar{\omega} - t\log 2)\|\mathbb{V}\| - \bar{\omega} d(\operatorname{supp} \mathbb{V}, V_1 \div V_2)]$$

whenever V_1, V_2 are cylinders, $V_1 \div V_2 = (V_1 \backslash V_2) \cup (V_2 \backslash V_1)$ is their symmetric difference, and $\mathbb{V} \in \mathscr{W}^{\mathrm{co,f}}(V_1 \cap V_2)$.

(c)
$$|\rho_V^{\mathscr{I}}(\mathbb{V}_1 \cup \mathbb{V}_2) - \rho_V^{\mathscr{I}}(\mathbb{V}_1)\,\rho_V^{\mathscr{I}}(\mathbb{V}_2)| \leqslant 3\exp[-(\bar{\omega} - t\log 2)\|\mathbb{V}_1 \cup \mathbb{V}_2\| - \tfrac{1}{2}\bar{\omega}\, d(\operatorname{supp} \mathbb{V}_1, \operatorname{supp} \mathbb{V}_2]$$

whenever $\mathbb{V}_1 \cup \mathbb{V}_2 \in \mathscr{W}^{\mathrm{co,f}}(V)$.

(ii) Similarly, denoting

$$\rho_{q,V}^{\mathrm{e}}(\theta) = P_{q,V}^{\mathrm{e}}(\{\bar{\theta} \in \mathscr{K}_q^{\mathrm{e}}(V) \mid \bar{\theta} \supset \theta\})$$

whenever $\theta \in \mathscr{K}_q^{\mathrm{e}}(V)$ and $V \subset \mathbb{Z}^\nu$, $q = 1, 2$, we have:

(a) $\rho_{q,V}^{\mathrm{e}}(\theta) \leqslant \exp(-\tau\|\theta\|)$

(b) $|\rho_{q,V_1}^{\mathrm{e}}(\theta) - \rho_{q,V_2}^{\mathrm{e}}(\theta)| \leqslant \exp[-\tau\|\theta\| - \omega d(\operatorname{supp} \theta, V_1 \div V_2)]$

whenever $V_1, V_2 \subset \mathbb{Z}^\nu$, $\theta \in \mathscr{K}_q^{\mathrm{e}}(V_1 \cap V_2)$.

(c)
$$|\rho_{q,V}^{\mathrm{e}}(\theta_1 \cup \theta_2) - \rho_{q,V}^{\mathrm{e}}(\theta_1)\,\rho_{q,V}^{\mathrm{e}}(\theta_2)| \leqslant \exp[-\tau\|\theta_1 \cup \theta_2\| - \omega d(\operatorname{supp} \theta_1, \operatorname{supp} \theta_2)]$$

whenever $\theta_1 \cup \theta_2 \in \mathscr{K}_q^{\mathrm{e}}(V)$.

Remarks. The probabilities $P_{q,V}^{\mathrm{e}}$ are actually the contour model probabilities corresponding to pure stable phases and constructed in the Pirogov–Sinai theory (see Section 3, and Proposition 3.4 in particular, for more details).

The most interesting case is when $V = \mathbb{Z}^\nu$. The formula (2.1) is also useful for cylinders V with finite base, since it may be combined with estimates (b) to control the speed of the convergence of $\mu_V^{\beta H}(\cdot \mid y)$ when $V \nearrow \mathbb{Z}^\nu$.

2.6. Surface Tension

To formulate a statement about the existence of (the thermodynamic limit of) surface tension (interfacial free energy), let us consider the cylinder $V_B = \{i \in \mathbb{Z}^\nu \mid (i_2,..., i_\nu) \in B\}$ when $B \subset \mathbb{Z}^{\nu-1}$ is finite.

Theorem 3. Let $\nu \geqslant 3$ and let β and $H \equiv \{\varphi_A\}$ satisfy the assumptions of Theorem 1 (or its Corollary). Then the limit

$$\sigma = \lim_{\substack{B \nearrow \mathbb{Z}^{\nu-1} \\ \text{van Hove}}} \frac{1}{|B|} \lim \text{fin cyl} \log$$

$$\times \frac{\{Z(U \mid y^{1,2}; \beta H)}{[Z([U \cap \mathbb{Z}_1^\nu(\mathrm{I}_0)] \backslash (I_0)_R \mid x^1; \beta H) \, Z([U \cap \mathbb{Z}_2^\nu(I_0)] \backslash (I_0)_R \mid x^2; \beta H)]^{-1}\}}$$

exists and is

$$\sigma = -\beta \sum_{\substack{A \subset (I_0)_R \\ A \cap x(\{i\}) \neq \varnothing}} \varphi_A(y) \frac{|\pi(\{i\})|}{|\pi(A)|} + \Delta$$

where $i \in \mathbb{Z}^\nu$ is arbitrary and Δ satisfies the inequality

$$|\Delta| \leqslant [\exp(-\bar{\omega}) + 2|\pi(i)| \exp(-\omega)]$$

with ω, $\bar{\omega}$ from Theorem 2.

Remarks.

1. The limit $B \nearrow \mathbb{Z}^{\nu-1}$ is considered in the van Hove sense, $|\partial B|/|B| \to 0$. An explicit formula for Δ is given in Proposition 7.1.

2. The case $\nu = 2$ has to be studied by slightly different means and it was considered in Ref. 6, where, supposing the existence (proven for ferromamgnets) of the limit, the inequality $|\sigma| \geqslant K\beta$ was proven.

In Ref. 6 a tacit assumption that the interaction $\{\varphi_A\}$ is reflection-invariant was used. It was needed, e.g., to prove

$$\sum_{X \subset V_1^0} \Phi_1^T(X) - \sum_{X \subset V_2^0} \Phi_2^T(X) = 0$$

in the formula (3.2) from Ref. 6.

3. We use here a different normalization than that used in Ref. 6; namely, we use the normalizing factor

$$Z(U_1(I_0) \backslash (I_0)_R \| x^1; \beta H) \cdot Z(U_2(I_0) \backslash (I_0)_R \mid x^2; H) \qquad (*)$$

instead of the factor (used in Ref. 4)

$$(Z(U|x^1;\beta H)\cdot Z(U|x^2;\beta H))^{1/2} \tag{$**$}$$

The difference between (*) and (**) is not essential in the case when both the finite cylinders and the interaction are invariant under reflections with respect to I_0. In the general case the factor (*) sems to be more suitable [than (**)], since it satisfies the natural requirement that the value of σ should not depend on the concrete "shape" of U.

3. REVIEW OF THE PIROGOV–SINAI THEORY

In this section we present notation and recall some statements of the Pirogov–Sinai theory[1,2] in a form used in our proofs.

3.1. Partition Functions and a Connection with Contour Models

First, it is easy to verify a connection between the "diluted relative" partition function

$$\theta(V|x^q;\beta H)=\sum_{\substack{x_{V^c}=x^q_{V^c}\\ B(x)\subset\subset V}}\exp\left\{-\beta\sum_{A}\left[\varphi_A(x)-\varphi_A(x^q)\right]\right\}$$

used by Pirogov and Sinai and the "physical" partition function $Z(V|x^q;\beta H)$ introduced in Section 2.1.

Lemma 3.1. Let $V\subset\mathbb{Z}^\nu$ be finite and $V_R=\{i|d(i,V)<R+1\}$. Then

$$\theta(V|x^q;\beta H)=\exp[\beta e(x^q)|V|]\sum_{\substack{x_{V^c}=x^q_{V^c}\\ B(x)\subset\subset V}}\exp[-\beta E_V(x)]$$

and

$$Z(V|x^q;\beta H)=\exp\left[\beta\sum_{A\subset V^c}\varphi_A(x^q)\frac{|A\cap V_R|}{|A|}-\beta e(x^q)|V_R|\right]\theta(V_R|x^q;\beta H)$$

If γ is a q-contour, Pirogov and Sinai introduce the "crystal" partition function of γ by

$$\theta(\gamma;\beta H)=\sum_{\theta(x)=\{\gamma\}}\exp\left\{-\beta\sum\left[\varphi_A(x)-\varphi_A(x^q)\right]\right\}$$

The following proposition summarizes the main statements of the Pirogov–Sinai theory. For a short review of the theory of contour models ("polymer models"), see Appendix B. In the following we use the results and notations from it in a substantial way. Note in particular that by a contour functional $\Phi(\gamma)$ we denote the weight ("fugacity") of the contour γ and, thus the partition function $\mathscr{Z}_q(V \mid \phi, b)$ of a contour model Φ with parameter $b \geqslant 0$ is defined as

$$\mathscr{Z}_q(V \mid \Phi, b) = \sum_{\partial \in \mathscr{K}_q^a(V)} \exp\left(b \left| \bigcup_{\gamma \in \theta(\partial)} \operatorname{Int} \gamma \right|\right) \Phi(\partial)$$

where $\Phi(\partial) = \prod_{\gamma \in \partial} \Phi(\gamma)$. By the constant c we again denote the constant from the estimate (B10) on the number of contours of a given length. Introducing the constant

$$c_3 = c_3(v, |S|, R) = 3 + \log(2c) + [v \log(2R+1)]/(2R+1)^v$$

we have the following result:

Proposition 3.2. Let $H \equiv \{\varphi_A\}$ satisfy the assumptions (I) and (CP) with respect to a collection of translation-invariant configurations $\{x^1,..., x^r\}$. Whenever $\beta\rho_1 \geqslant c_3$, there exists for every $q = 1,..., r$ a contour functional Φ_q and a parameter $b_q \geqslant 0$ such that:

(i) Φ_q is a τ-functional, $|\Phi_q(\gamma)| \leqslant e^{-\tau|\gamma|}$ for each $\gamma \in \mathbb{K}_q$, with $\tau = \beta\rho_1 - 2$.

(ii) For each $\gamma \in \mathbb{K}_q$ one has

$$\theta(\gamma; \beta H) = [\exp(b_q |\operatorname{Int} \gamma|)] \, \Phi_q(\gamma) \, \mathscr{Z}_q(\operatorname{Int} \gamma; \Phi_q)$$

and (thus) also

$$\theta(V \mid x^q; \beta H) = \mathscr{Z}_q(V; \Phi_q, b_q)$$

for each finite $V \subset \mathbb{Z}^v$.

(iii) $\min_{q=1,...,r} b_q = 0$.

(iv) The limit $p(\beta H) = \lim[\log Z(V \mid x; \beta H)]/|V|$, with $V \nearrow \mathbb{Z}^v$ in the van Hove sense, exists for every $x \in X$ (and does not depend on x) and

$$b_q - \beta e(x^q) + p(\Phi_q) = p(\beta H)$$

for each $q = 1,..., r$.

(v) Denoting

$$\psi_q(U) = \log \Theta(V \mid x^q; \beta H) - [\beta e(x^q) + p(\beta H)] \, |U|$$

we have

$$|\psi_q(U)| \leqslant \{\exp[-\omega(2R+1)^\nu]\}^{|\partial U|}$$

with $\omega = \beta\rho_1 - c_3$ whenever $U \subset \mathbb{Z}^\nu$ and $q \in \{1,..., r\}$.

Proof. For (i), (ii), and the equality $b_q - \beta e(x^q) + p(\Phi_q) = \alpha$ with α such that $\min_{q=1,...,r} b_q = 0$ see Refs. 1 and 2. For the computation of τ from (i) we used the version of Ref. 13 combined with estimates (B12) and (B4′) from Theorem B.2. Considering then q with $b_q = 0$ and observing that the limit

$$\lim \frac{\log \mathscr{Z}_q(V; \Phi_q)}{|V|} = p(\Phi_q)$$

exists [Theorem B.2(iv)], the existence of the limit in (iv) as well as the equality $\alpha = p(\beta H)$ follow from Lemma 3.1. The statement (v) plays an important role in this theory. It folows from the inequalities

$$\begin{aligned}
&\theta(V \mid x^q; \beta H) \exp[-\beta e(x^q)|U| - p(\beta H)|U|] \\
&\quad = \mathscr{Z}_q(U; \Phi_q, b_q) \exp\{-[\beta e(x^q) + p(\beta H)]|U|\} \\
&\quad \leqslant \mathscr{Z}_q(U; \Phi_q) \exp[(-\beta\beta e(x^q) - p(\beta H) + b_q)|U|] \\
&\quad \leqslant \exp[-\beta e(x^q) + b_q + p(\Phi_q) - p(\beta H) + d \cdot |\partial U|] \\
&\quad = \exp(d \cdot |\partial U|)
\end{aligned}$$

where we used the key estimate

$$\mathscr{Z}_q(U; \Phi_q) \leqslant \exp[p(\Phi_q)|U| + d|\partial U|]$$

with $d = \exp[-\omega(2R+1\}^\nu]$ following from Theorem B.2(iv).

The Pigorov–Sinai theory also contains a statement about the full phase diagram in a neighborhood of a Hamiltonian H_0 with translation-invariant ground configurations $x^1,..., x^r$.

A configuration $x \in X$ is called a *ground configuration* of H if $\sum_A [\varphi_A(z) - \varphi_A(x)] \geqslant 0$ whenever z is a configuration differing from x only in a finite $V \subset \mathbb{Z}^\nu$: $x_{V^c} = z_{V^c}$. Consider now a Hamiltonian $H_0 \equiv \{\varphi_A^0\}$ satisfying (I) and a set of translation-invariant configurations $\{x^1,..., x^r\}$ such that every x^q, $q = 1,..., r$, is a ground configuration of H_0. Let further $H_s \equiv \{\varphi_A^s\}$, $s = 1,..., r-1$, be additional Hamiltonians ("external fields") fulfilling (I) such that the Hamiltonian $H_\mu = H_0 + \sum \mu_s H_s$, $\mu = (\mu_1,..., \mu_{r-1}) \in \mathbb{R}^{r-1}$, *completely removes the degeneracy* of ground configuration of H_0. Namely, denoting by $e_\mu(x^q)$ the specific energy of x^q with respect to H_μ, we have that the mapping

$$\mu \to e_\mu(x^q) - \min_{m=1,..., r} e_\mu(x^m)$$

maps the space of parameters $\mathbb{R}^{r-1}$ onto the entire boundary

$$O_r = \{b \mid b = (b_1,..., b_r), \min_{q=1,...,r} b_q = 0\}$$

of the r-dimensional positive octant. Recalling that

$$\|H\| = \sup_{i \in \mathbb{Z}^\nu} \sup_{x \in X} |E_{\{i\}}(x)|$$

we have the following result.

Proposition 3.3. Let H_0 satisfy the assumptions (I) and (CP) with respect to a collection of its translation-invariant ground configurations $\{x_1,..., x_r\}$ and let H_s, $s = 1,..., r-1$, fulfilling (I) be such that $H_\mu = H_0 + \sum \mu_s H_s$ completely removes the degeneracy. Let us denote $K = \sup_{s=1,...,r-1} \|H_s\|$ and consider $\varepsilon < \rho_1/2K$ and $\beta \geqslant (c_3)/(\rho_1 - 2K\varepsilon)$. Whenever $\mu \in U_0(\varepsilon) = \{\mu \in \mathbb{R}^{r-1} \mid \sum |\mu_s| < \varepsilon\}$, the Hamiltonian H_μ satisfies the assumptions of Proposition 3.2. Then, denoting $\{b_1(\mu),..., b_r(\mu)\}$ the corresponding parameters, the mapping $\mu \to \{b_1(\mu),..., b_r(\mu)\}$ is a homeomorphism of $U_0(\varepsilon)$ into O_r such that the image of U_0 contains a neighborhood of $O \in O_r$. Moreover, to every $b_q(\mu) = 0$ there corresponds an extremal translation-invariant Gibbs state of H_μ (at the inverse temperature β); the number of all different extremal periodic Gibbs states of H_μ equals the number of vanishing parameters $b_q(\mu)$.

Proof. If H_0 satisfies (CP), then $H_0 + \sum \mu_s H_s$ satisfies (CP) with the constant $\rho = \rho_1 - 2K(\sum |\mu_s|) \geqslant \rho_1 - 2K\varepsilon$ whenever $\mu \in U_0(\varepsilon)$. Then one uses Proposition 3.2 and follows the proof of Main Theorem B in Ref. 1. For a proof that the set of extremal Gibbs states corresponding to $b_q = 0$ exhausts the set of all periodic extremal Gibbs states see Ref. 14.

3.2. Description of Stable Phases

We shall use a more detailed description of *stable phases*, i.e., Gibbs states corresponding to vanishing parameters $b_q = 0$. The following statement is essentially contained in Ref. 1 and especially in Ref. 2. An explicit expression of the form (3.1) appears in Ref. 15.

Proposition 3.4. Let the assumptions o Proposition 3.2 be fulfilled and let $q \in \{1,..., r\}$ be such that $b_q = 0$ and $V \subset \mathbb{Z}^\nu$. Then there exists a Gibbs state μ in V and a probability measure P^{e}_{q, V_R} on $\mathscr{K}^{\mathrm{e}}_q(V_R)$ such that for every bounded, measurable φ one has

$$\mu(\varphi) = \int_{\mathscr{K}^{\mathrm{e}}_q(V_R)} \mu(\varphi \mid \theta)\, P^{\mathrm{e}}_{q, V_R}(d\theta) \tag{3.1}$$

where $\mu(\cdot|\theta)$ is the unique Gibbs state in $J^0(\theta)$ under the boundary condition $x_{J^0(\theta)^c}$, where

$$J^0(\theta)=\left\{i\in\mathbb{Z}^\nu \mid d\left[i,\left(\bigcup_{\gamma\in\theta}\operatorname{Int}\gamma\right)^c\right]>R+1\right\}$$

and x is such that $\theta(x)=\theta$. Moreover:

(i) μ is a weak limit of $\mu_U^{\beta H}(\cdot|x^q)$ over finite $U\subset V$, ordered by inclusion.

(ii) Denoting $\mathscr{K}_q^{\mathrm{e}}(\theta, V) = \{\bar\theta \in \mathscr{K}_q^{\mathrm{e}}(V) \mid \bar\theta \supset \theta\}$ and $\rho_{q,V}^{\mathrm{e}}(\theta) = P_q^V(\mathscr{K}_q^2(\theta, V))$ whenever $\theta\in\mathscr{K}_q^{\mathrm{e}}(V)$, and taking $\tau=\beta\rho_1-2$ and

$$\omega=\beta\rho_1-c_3-\frac{\nu\log(2R+1)}{(2R+1)^\nu}$$

we have:

(a) $\rho_{q,V}^{\mathrm{e}}(\theta)\leqslant e^{-\tau\|\theta\|}$

for every $\theta\in\mathscr{K}_q^{\mathrm{e}}(V)$.

(b) $|\rho_{q,V_1}^{\mathrm{e}}(\theta)-\rho_{q,V_2}^{\mathrm{e}}(\theta)|$

$$\leqslant\|\theta\|\exp[-\tau\|\theta\|-\omega d(\operatorname{supp}\theta, V_1\div V_2)]$$

for every $V_1, V_2\subset\mathbb{Z}^\nu$ and $\theta\in\mathscr{K}_q^{\mathrm{e}}(V_1\cap V_2)$.

(c) $|\rho_{q,V}^{\mathrm{e}}(\theta_1\cup\theta_2)-\rho_{q,V}^{\mathrm{e}}(\theta_1)\,\rho_{q,V}^{\mathrm{e}}(\theta_2)|$

$$\leqslant\|\theta_1\cup\theta_2\|\exp[-\tau\|\theta_1\cup\theta_2\|-\omega d(\operatorname{supp}\theta_1, \operatorname{supp}\theta_2)]$$

whenever $\theta_1\cup\theta_2\in\mathscr{K}_q^{\mathrm{e}}(V)$.

(iii) There exist $\alpha>0$ and $K>0$ such that

(a) $|\mu_{V_1}(\varphi)-\mu_{V_2}(\varphi)|\leqslant K|\Lambda|\,\|\varphi\|\exp[-\alpha d(\Lambda, V_1\div V_2)]$

whenever φ is a cylinder function living in Λ [i.e., $\varphi(x)=\varphi(y)$ whenever $x_\Lambda=y_\Lambda$].

(b) $|\mu_V(\varphi_1\varphi_2)-\mu_V(\varphi_1)\,\mu_V(\varphi_2)|$

$$\leqslant K|\Lambda_1\cup\Lambda_2|\,\|\varphi_1\|\,\|\varphi_2\|\exp[-\alpha d(\Lambda_1, \Lambda_2)]$$

whenever φ_1, φ_2 are cylinder functions living in Λ_1, Λ_2, respectively.

Proof. Let ϕ_q be the τ-functional from Proposition 3.2. According to Theorem B.2(iii), there exists a measure P_{q,V_R} on $\mathscr{K}_q^{\mathrm{a}}(V_R)$ that recovers its correlation functions $\rho_{V_R}(\partial\mid\Phi_q)$. Introducing a map $\mathscr{K}_q^{\mathrm{a}}(V_R)\to\mathscr{K}_q^{\mathrm{e}}(V_R)$ by attributing to each $\partial \in \mathscr{K}_q^{\rm a}(V_R)$ the set of its external contours $\theta(\partial) \in \mathscr{K}_q^{\rm e}(V_R)$, we may define the measure $P^{\rm e}_{q,V_R}$ on $\mathscr{K}_q^{\rm e}(V_R)$ as the image of P_{q,V_R} under this map. Let us observe that for finite U one has

$$\rho^{\rm e}_{q,U}(\theta) = \Phi_q(\theta)\,\frac{\mathscr{Z}(\mathbb{K}_q(U)\setminus[[\theta]])}{\mathscr{Z}(\mathbb{K}_q(U))}$$

$$= \Phi_q(\theta) \exp\left[-\sum_{\mathbb{C} \in \mathscr{K}_q^{\rm cl}(U);\, \mathbb{C} \cap [[\theta]] \neq \varnothing} \Phi_q^T(\mathbb{C})\right]$$

where $[[\theta]] = \{\gamma \in \mathbb{K}_q \mid \text{ either } \gamma \iota \theta \text{ or there exists } \bar{\gamma} \in \theta \text{ such that } \operatorname{supp} \bar{\gamma} \subset \operatorname{Int} \gamma\}$ (see Appendix B). Taking into account the estimate (B4′) and the fact that

$$\|C\| \leqslant \exp\left[\frac{\nu \log(2R+1)}{(2R+1)^\nu}\,\|C\|\right]$$

since $\|C\| \geqslant (2R+1)^\nu$, we get the bound

$$\sum_{\bigcup_{\gamma \in \mathbb{C}} (\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \ni i} |\Phi_q^T(\mathbb{C})|\, e^{\omega \|\mathbb{C}\|} \leqslant \sum_{\operatorname{supp} \mathbb{C} \ni i} |\Phi_q^T(\mathbb{C})|\, \|\mathbb{C}\|\, e^{\omega \|\mathbb{C}\|} \leqslant 1$$

by similar reasoning as when proving (B.13).

Hence, taking into account the positivity of $\Phi_q(\theta)$ and the inequality $|e^u - e^v| \leqslant \max(e^u, e^v)\,|u - v|$, one easily verifies (ii).

To prove (i) and (3.1), let us consider a cylinder function φ living in $\Lambda \subset \mathbb{Z}^\nu$ and choose $\varepsilon > 0$. We shall prove that for $U \subset V$ finite and large enough,

$$\left| \mu_U^{\beta H}(\varphi \mid x^q_{U^c}) - \int_{\mathscr{K}_q^{\rm e}(V_R)} \mu(\varphi \mid \theta)\, P^{\rm e}_{q,V_R}(d\theta) \right| \leqslant \varepsilon \|\varphi\| \tag{3.2}$$

From Proposition 3.2(ii), Lemma 3.1, and the fact that $b_q = 0$, one easily verifies that if $U \subset \mathbb{Z}^\nu$ is finite, then

$$\mu_U^{\beta H}(\varphi \mid x^q_{U^c}) = \sum_{\theta \in \mathscr{K}_q^{\rm e}(U_R)} \mu(\varphi \mid \theta)\,\frac{\mathscr{Z}(\theta; \Phi_q)}{\mathscr{Z}_q(U_R; \Phi_q)}$$

$$\equiv \int_{\mathscr{K}_q^{\rm e}(U_R)} \mu(\varphi \mid \theta)\, P^{\rm e}_{q,U_R)}\, \mu(\varphi \mid \theta)\, P^{\rm e}_{q,U_R}(d\theta)$$

Hence, to prove (3.2) means proving

$$\left| \int_{\mathscr{K}_q^{\rm e}(U_R)} \mu(\varphi \mid \theta)\, P^{\rm e}_{q,U_R}(d\theta) - \int_{\mathscr{K}_q^{\rm e}(V_R)} \mu(\varphi \mid \theta)\, P^{\rm e}_{q,V_R}(d\theta) \right| \leqslant \varepsilon \|\varphi\| \tag{3.3}$$

for U large enough. Whenever $\theta \in \mathscr{K}_q^e$, we shall consider a subset $\theta^{(k)} \subset \theta$ of those $\gamma \subset \theta$ for which $|\gamma| < k$. Denoting by $\mathscr{K}_q^e(\Lambda, k)$ the set $\{\theta \in \mathscr{K}_q^e \mid$ there exists $\gamma \in \theta$ such that $\Gamma \cap \Lambda \neq \varnothing$ and $|\gamma| > k\}$, we get, using (ii)(a), the estimate

$$\int [\mu(\varphi \mid \theta) - \mu(\varphi \mid \theta^{(k)})] \, P_{q,U}^e(d\theta) \leqslant 2 \, \|\varphi\| \, P_{q,U}^e(\mathscr{K}_q^e(\Lambda, k))$$

$$\leqslant 2 \, \|\varphi\| \, |\Lambda| \, e^{-\tau k} \leqslant \tfrac{1}{4}\varepsilon \, \|\varphi\|$$

whenever $U \subset \mathbb{Z}^\nu$, and k is large enough. Having chosen such k, the estimate (3.3) will be verified if we show that

$$\left| \int_{\mathscr{K}_q^e(U_R)} \mu(\varphi \mid \theta^{(k)}) \, P_{q,U_R}^e(d\theta) - \int_{\mathscr{K}_q^e(V_R)} \mu(\varphi \mid \theta) \, P_{q,V_R}^e(d\theta) \right| \leqslant \frac{1}{2}\varepsilon \, \|\varphi\| \quad (3.4)$$

for U large enough. Observing that $\mu(\varphi \mid \theta^{(k)})$ is a cylindrical function living in $\Lambda_k = \{i \in \mathbb{Z}^\nu \mid d(i, \Lambda) < k\}$ [i.e., if $\theta_1, \theta_2 \in \mathscr{K}_q^e$ are such that $\theta_1 \cap \mathbb{K}_q(\Lambda_k) = \theta_2 \cap \mathbb{K}_q(\Lambda_k)$, then $\mu(\varphi \mid \theta_1^{(k)}) = \mu(\varphi \mid \theta_2^{(k)})$], the estimate (3.4) follows from the weak convergence $\lim_{U \nearrow V} P_{q,U_R}^e = P_{q,V_R}^e$ [Theorem B.2(iii)]. According to (3.2), thus $\lim_{U \nearrow V} \mu_U^{\beta H}(\varphi \mid x_{U^c}^q)$ exists and is equal to

$$\int_{\mathscr{K}_q^e(V_R)} \mu(\varphi \mid \theta) \, P_{q,V_R}^e(d\theta)$$

It is clearly a Gibbs state in V. Validity of (3.1) for all measurable, bounded φ then follows from the fact that both sides of (3.1) have unique extensions from bounded cylindrical functions to bounded measurable functions.

Finally, to prove (iii), we first realize that replacing $\rho_{q,V}^e(\theta)$ in (ii) by $\rho_{q;V;\Lambda}^e(\theta) = P_q^V(\mathscr{K}_q^e(\theta, V; \Lambda))$ with

$$\mathscr{K}_q^e(\theta, V; \Lambda) = \{\bar{\theta} \in \mathscr{K}_q^e(\theta, V) \mid \gamma \in \bar{\theta} \backslash \theta \text{ implies } (\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \cap \Lambda = \varnothing\}$$

we get similar estimates:

(a′) $\rho_{q,V;\Lambda}^e(\theta) \leqslant \exp(-\tau \, \|\theta\|)$

(b′) $|\rho_{q;V_1;\Lambda}^e(\theta) - \rho_{q;V_2;\Lambda}^e(\theta)|$

$$\leqslant \|\theta\| \max(\rho_{q,V_1;\Lambda}^e(\theta), \rho_{q,V_2;\Lambda}^e(\theta)) \times \exp[-\omega d(\operatorname{supp} \theta \cup V, V_1 \div V_2)]$$

(c′) $|\rho_{q,V;\Lambda_1 \cup \Lambda_2}^e(\theta_1 \cup \theta_2) - \rho_{q,V;\Lambda_1}^e(\theta_1) \, \rho_{q,V;\Lambda_2}^e(\theta_2)|$

$$\leqslant \|\theta_1 \cup \theta_2\| \max(\rho_{q,V;\Lambda_1 \cup \Lambda_2}^e(\theta_1 \cup \theta_2), \rho_{q,V;\Lambda_1}^e(\theta_1) \, \rho_{q,V;\Lambda_2}^e(\theta_2)) \times \exp[-\omega d(\Lambda_1 \cup \operatorname{supp} \theta_1, \Lambda_2 \cup \operatorname{supp} \theta_2)]$$

To prove (iii)(a), we notice that $P_q^V(\mathscr{K}_q^{\mathrm{e}}(\Lambda, k)) \leqslant |\Lambda| e^{-\omega k}$ and realizing that $\theta \in \mathscr{K}_q^{\mathrm{e}} \backslash \mathscr{K}_q^{\mathrm{e}}(\Lambda, k)$ such that $(\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \cap \Lambda \neq \varnothing$ for each $\gamma \in \theta$ implies $\|\theta\| \leqslant |\Lambda| k^{\nu}$, we get

$$\begin{aligned}
&|\mu_{V_1}(\varphi) - \mu_{V_2}(\varphi)| \\
&\quad \leqslant 2\, \|\varphi\|\, |\Lambda| \exp(-\omega k) \\
&\qquad + \sum_{\substack{\theta \in \mathscr{K}_q^{\mathrm{e}} \backslash \mathscr{K}_q^{\mathrm{e}}(\Lambda, k) \\ \gamma \in \theta \Rightarrow (\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \cap \Lambda \neq \varnothing}} \mu(\varphi \mid \theta) |\rho_{q, V_1; \Lambda}^{\mathrm{e}}(\theta) - \rho_{q, V_2, \Lambda}^{\mathrm{e}}(\theta)| \\
&\quad \leqslant 2\, \|\varphi\|\, |\Lambda| \exp(-\omega k) + \|\varphi\|\, |\Lambda|\, k^{\nu} \exp\{-\omega[d(\Lambda, V_1 \div V_2) - 2k]\} \\
&\qquad \times \sum_{\substack{\theta \in \mathscr{K}_q^{\mathrm{e}} \backslash \mathscr{K}_q^{\mathrm{e}}(\Lambda, k) \\ \gamma \in \theta \Rightarrow (\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \cap \Lambda \neq \varnothing}} [\rho_{q, V_1; \Lambda}^{\mathrm{e}}(\theta) + \rho_{q, V_2; \Lambda}^{\mathrm{e}}(\theta)] \\
&\quad \leqslant \|\varphi\|\, |\Lambda| (2 \exp(-\omega k) + 2k^{\nu} \exp\{-\omega[d(\Lambda, V_1 \div V_2) - 2k]\})
\end{aligned}$$

The last inequality follows upon realizing that the sets $\mathscr{K}_q^{\mathrm{e}}(\theta, V; \Lambda)$ are disjoint for different $\theta \in \mathscr{K}_q^{\mathrm{e}} \backslash \mathscr{K}_q^{\mathrm{e}}(\Lambda, k)$ such that $(\operatorname{supp} \gamma \cup \operatorname{Int} \gamma) \cap \Lambda \neq \varnothing$ for every $\gamma \in \theta$. Taking $k = \frac{1}{3} d(\Lambda, V_1 \div V_2)$, we get the desired estimate.

The estimate (iii)(b) is proved in a similar way, using (ii)(c′).

4. PROBABILISTIES OF INTERFACES IN TERMS OF A CONTOUR MODEL

In this section we introduce certain new partition functions $\tilde{Z}$ obtained from Z by dividing it by suitable "normalizing" factors. The advantage of these new partition functions will be the possibility of rewriting them in a form very close to that one used in contour models, such, moreover, that the "contours" [these will be defined as some "aggregates" of walls and clusters of 1- (2-) contours] will "live" near I_0, i.e., an essentially $(\nu - 1)$-dimensional contour model will be obtained.

In Section 4.1, we define the normalized partition functions $\tilde{Z}$ and rewrite them in Lemma 4.4. In Section 4.2, we pass to an infinite cylindrical volume. We define the notion of an aggregate and formulate and prove Lemma 4.9.

4.1. Expression in Terms of Walls

In this subsection $V \subset \mathbb{Z}^{\nu}$ will always be a finite volume. In Appendix A we prove the following result:

Lemma 4.1. Let $\nu \geqslant 2$ and let $x_i = y_i$ except for finitely many $i \in \mathbb{Z}^{\nu}$. Then x has an interface.

Note that whenever $\mathbb{I}=(I, x_I)$ is an interface of a configuration x that differs from y only on V, then $\mathbb{I}\in\mathscr{I}(V)$ (cf. Section 2.5). Let us denote by $\mathrm{Int}(I)$ the union of finite components $\mathrm{Int}_\alpha(I)$ of the complement of $\mathrm{supp}(\mathbb{I})$ and recall that by $V_1(I)$, $V_2(I)$ we denoted the intersection of $\mathbb{Z}_1^\nu(I)$, $\mathbb{Z}_2^\nu(I)$, respectively, with V_R. We note also that for any α there is $m(\alpha)\in\{1,..., r\}$ such that $x_i=x_i^{m(\alpha)}$ whenever $i\in\mathrm{Int}_\alpha(I)$ and $d(i, I)\leqslant R+1$ or $i\in I$ and $d(i, \partial\,\mathrm{Int}_\alpha(I))\leqslant R-1$. It follows from this observation that $E_U(x)$ depends only on x_I whenever x is a configuration such that $\mathbb{I}(x)=(I, x_I)$ and $U\subset I$. We use the notation $E_U(\mathbb{I})$ in such cases. Suppressing βH in the following notation, we have the following result:

Lemma 4.2. $Z(V\mid y)=\sum_{\mathbb{I}\in\mathscr{I}(V)} Z(\mathbb{I}, V\mid y)$, where

$$Z(\mathbb{I}, V\mid y)=\exp\left[-\beta E_{I\cap V_R}(\mathbb{I})+\beta\sum_{A\subset V^c}\varphi_A(y)\frac{|A\cap V_R|}{|A|}\right.$$
$$\left.-\sum_{q=1}^{2}\beta e(x^q)\,|V_q(I)|\right]\prod_{q=1}^{2}\theta(V_q(I)\mid x^q)$$
$$\times\prod_\alpha\exp[-\beta e(x^{m(\alpha)})\,|\mathrm{Int}_\alpha(I)|\,]\;\theta(\mathrm{Int}_\alpha(I)\mid x^{m(\alpha)})$$

Proof. We choose an interface $\mathbb{I}=(I, x_I)\in\mathscr{I}(V)$ and consider any x with $\mathbb{I}=\mathbb{I}(x)$. One proves Lemma 4.2 easily using the equalities

$$H_V(x)=E_{V_R}(x)-\sum_{\substack{A\subset V^c\\ A\cap V_R\neq\varnothing}}\varphi_A(x)\frac{|A\cap V_R|}{|A|}$$
$$=E_{I\cap V_R}(x)+E_{V_1(I)}(x)$$
$$+E_{V_2(I)}(x)+E_{\mathrm{Int}(I)\cap V_R}(x)-\sum_{\substack{A\subset V^c\\ A\cap V_R\neq\varnothing}}\varphi_A(x)\frac{|A\cap V_R|}{|A|}$$

and Lemmas 3.1 and 4.1.

Now we shall extract from $Z(\mathbb{I}, V\mid y)$ some terms that do not depend on the interface $\mathbb{I}$.

Let us denote $V_{q0}=V_q(\mathbb{I}(y))$ and $\partial^q V_R=\partial V_R\cap\partial V_{q0}$, $q=1, 2$. If $\mathbb{C}$ is a cluster, we always write C instead of $\mathrm{supp}(\mathbb{C})$ (see Appendix B for corresponding definitions). We write $C_* q$ if there are $i, j\in C$ such that $|i-j|=1$, $i\in\partial^q V_R$, $j\in V_R^c$. Let us put $\chi_q(C)=1$ if $C_* q$ and $\chi_q(C)=0$ otherwise. Let $\mathbb{I}_0=(I_0, y_{I_0})=(I(y)\; y_{I(y)})$.

Before continuing in the expression of the partition sum we should emphasize that we suppose that the assumptions (I), (CP), (WP), and (S) are satisfied. Since we shall rely on the Pirogov–Sinai theory, we always suppose that the assumptions of Proposition 3.2 are fulfilled. In particular, we suppose that the inequality $\beta\rho_1 \geqslant c_3$ holds. The following lemma is based on the fundamental expression (B2):

Lemma 4.3. If $\mathbb{I} \in \mathscr{I}(V)$, one has

$$\begin{aligned}\frac{Z(\mathbb{I}, V\mid y)}{N(V\mid y)} &= \tilde{Z}(\mathbb{I}, V\mid y)\\ &= \exp\Big\{-p(\beta H)(|I\cap V_R| - |I_0 \cap V_R|)\\ &\quad -\beta[E_{I\cap V_R}(\mathbb{I}) - E_{I_0\cap V_R}(\mathbb{I}_0)] + \sum_\alpha \Psi_{m(\alpha)}(\mathrm{Int}_\alpha(I))\\ &\quad - \sum_{q=1}^{2} \sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ C\cap I\neq\varnothing}} \Phi_q^T(\mathbb{C})\left[\frac{|C\cap V_q|}{|C|} - \chi_q(C)\frac{|C\cap V_R|}{|C|}\right]\Big\}\end{aligned}$$

where

$$N(V\mid y) = N_v(V\mid y)\cdot N_{I_0}(V\mid y)\cdot N_s(V\mid y)$$

with the "volume term"

$$N_v(V\mid y) = \exp[p(\beta H)\,|V|]$$

the "surface term"

$$\begin{aligned}N_s(V\mid y) = \exp\Big[&p(\beta H)\,|V_R\setminus V| + \beta\sum_{A\subset V^c}\varphi_A(y)\frac{|A\cap V_R|}{|A|}\\ &- \sum_{q=1}^{2}\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ C_* q}} \Phi_q^T(\mathbb{C})\frac{|C\cap V_R|}{|C|}\Big]\end{aligned}$$

and the term extracted from Z to get the comparison with the flat interface $\mathbb{I}_0$ is

$$N_{\mathbb{I}_0}(V\mid y) = \exp[-p(\beta H)|\,\mathbb{I}_0\cap V_R| - \beta E_{I_0\cap V_R}(\mathbb{I}_0)]$$

Proof. One substitutes $\sum_{\mathbb{C}:C\subset V_q(I)}\Phi_q^T(\mathbb{C})$ for $\log\theta(V_q(I)\mid x^q)$ according to Proposition 3.2(ii) and (B2),

$$\begin{aligned}&\left[p(\beta H) - \sum_{\mathbb{C}:C\ni i}\frac{\Phi_q^T(\mathbb{C})}{|C|}\right]|V_q(i)|\\ &\qquad = p(\beta H)\,|V_q(I)| - \sum_{\mathbb{C}:C\cap V_q\neq\varnothing}\frac{\Phi_q^T(\mathbb{C})\,|C\cap V_q|}{|C|}\end{aligned}$$

for $-\beta e(x^q)|V_q(I)|$ according to Proposition 3.2(iv) and (B11), and

$$\exp[\psi_{m(\alpha)}(\mathrm{Int}_\alpha(I))]$$

for

$$\theta(\mathrm{Int}_\alpha(I)\,|\,x^{m(\alpha)})\exp[-\beta e(x^{m(\alpha)})\,|\mathrm{Int}_\alpha(I)| + p(\beta H)\,|\mathrm{Int}_\alpha(I)|]$$

according to Proposition 3.2(iv).

We use the notation

$$E(\mathbb{w}) = \beta(E_W(\mathbb{w}) - E_{\pi(W)}(\mathbb{I}_0)) + p(\beta H)(|W| - |\pi(W)|) + \sum_\alpha \psi_{m(\alpha)}(\mathrm{Int}_\alpha(W))$$

The notation $\mathrm{Int}_\alpha(W)$ stands for $\mathrm{Int}_\alpha(I)$ if $\alpha \in N(W) = \{\alpha \,|\, \mathrm{Int}_\alpha(I)$ is a finite component of $W^c\}$ whenever W is the support of a wall of $\mathbb{I}$ with the support I. It follows from the geometrical structure of walls (Lemma A.3) that $\{N(W)\}$ form a disjoint decomposition of the set of all α used as indices in the notation of components $\mathrm{Int}_\alpha(I)$ of $\mathrm{Int}(I)$. Using this notation, one immediately gets from Lemma 4.3 the following result:

Lemma 4.4.

$$\tilde{Z}(\mathbb{I}, V\,|\,y) = \prod_{\mathbb{w} \in \mathbb{W}(\mathbb{I})} \exp[-E(\mathbb{w})] \exp\left\{-\sum_{q=1}^{2} \sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\mathrm{cl}} \\ C \cap I \neq \varnothing}} \Phi_q^T(\mathbb{C}) \times \left[\frac{|C \cap V_q|}{|C|} - \chi_q(C)\frac{|C \cap V_R|}{|C|}\right]\right\}$$

for an $\mathbb{I} = (I, \mathbb{w}_I) \in \mathscr{I}(V)$.

To be more precise, we notice that $W \subset V_R$ for a wall of an interface $\mathbb{I} \in \mathscr{I}(V)$ and therefore

$$\begin{aligned} E(\mathbb{w}) = {} & \beta(E_{W \cap V_R}(\mathbb{I}) - E_{\pi(W) \cap V_R}(\mathbb{I}_0)) \\ & + p(\beta H)(|W \cap V_R| - |\pi(W) \cap V_R|) \\ & + \sum_\alpha \psi_{m(\alpha)}(\mathrm{Int}_\alpha(W) \cap V_R) \end{aligned}$$

and then use Lemma 4.3.

4.2. Expression in Terms of Families of Standard Aggregates

Now we pass from the case of a finite volume to an investigation of a cylindrical volume V with a finite basis $B \subset \mathbb{Z}^\nu$. Let us note that since $\mu_V^{\beta H}(\cdot\,|\,y)$ is unique [Lemma 2.1(ii)], we have

$$\lim_{U \nearrow V} \mu_U^{\beta H}(\cdot\,|\,y) = \mu_V^{\beta H}(\cdot\,|\,y)$$

where we consider the weak limit over the directed set of finite volumes $U \subset V$.

Let us consider a finite, nonempty $U \subset V$.

Lemma 4.5. Let $\beta\rho_1 \geqslant c_3$ and $\beta\rho_2 \geqslant c_4 = c_4(v, |S|, R) = 3^v + 5 + \log(2c)$. Then:

(a) There exists $K_B: \mathscr{I}(V) \to \mathbb{R}$ such that $\tilde{Z}(\mathbb{I}, U \mid y) \leqslant K_B(I)$ for any $\mathbb{I} \in \mathscr{I}(U)$ with $\sum_{\mathbb{I} \in \mathscr{I}(V)} K_B(I) < \infty$.

(b) There exists a finite limit $\operatorname{limfin\,cyl}_{U \nearrow V} \tilde{Z}(\mathbb{I}, U \mid y)$ and it equals

$$\tilde{Z}(\mathbb{I}, V \mid y) = \exp\left\{ - \sum_{\mathbb{w} \in \mathbb{W}(\mathbb{I})} E(\mathbb{w}) - \sum_{q=1}^{2} \sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\mathrm{cl}} \\ C \cap I \neq \varnothing}} \Phi_q^T(\mathbb{C}) \left[\frac{|C \cap V|}{|C|} - \chi_q(C) \frac{|C \cap V_R|}{|C|} \right] \right\}$$

where V_q, χ_q are defined as before.

(c) The probabilities $P_U^{\mathscr{I}}$ of interfaces from $\mathscr{I}(U)$ defined by

$$\mu_U^{\beta H}(\{x \mid \mathbb{I}(x) = \mathbb{I}\} \mid y) = P_U^{\mathscr{I}}(\mathbb{I}) = \frac{\tilde{Z}(\mathbb{I}, U \mid y)}{\sum_{\mathbb{I}' \in \mathscr{I}(U)} \tilde{Z}(\mathbb{I}', U \mid y)}$$

converge to a probability on $\mathscr{I}(V)$ (to be denoted by $P_V^{\mathscr{I}}$) which is defined by $\tilde{Z}(\mathbb{I}, V \mid y)$, i.e.,

$$P_V^{\mathscr{I}}(\mathbb{I}) = \frac{\tilde{Z}(\mathbb{I}, V \mid y)}{\sum_{\mathbb{I}' \in \mathscr{I}(V)} \tilde{Z}(\mathbb{I}', V \mid y)}$$

Proof. We see from Proposition 3.2(iv) that for any $q \in \{1, ..., r\}$

$$E(\mathbb{w}) = \beta(E_W(\mathbb{w}) - E_{\pi(W)}(\mathbb{I}_0) - e(x^q)[|W| - |\pi(W)|] + \sum_{\alpha} \psi_{m(\alpha)}(\operatorname{Int}_\alpha(W)) + [p(\Phi_q) + b_q][|W| - |\pi(W)|] \tag{4.1}$$

Let us notice that, according to Proposition 3.2(i), the assumptions of Theorem B.2 are satisfied for $\beta\rho_1 > 3 + \log(2c)$. Thus, we can use the estimates (B11) and (B4′) to get $|p(\phi_q)| \leqslant 1$. According to Proposition 3.2(iv), the sum $\sum_\alpha |\psi_{m(\alpha)}(\operatorname{Int}_\alpha(W))|$ can be estimated by $\sum_\alpha |\partial \operatorname{Int}_\alpha(W)|$. Since for each element i of $\partial \operatorname{Int}_\alpha(W)$ at least one of its $3^v - 1$ neighbors belongs to W, we have $\sum_\alpha |\partial \operatorname{Int}_\alpha(W)| \leqslant 3^v |W|$.

If we consider the cases $|W| - |\pi(W)| \geqslant 0$ and $\leqslant 0$ separately, we get [considering in (4.1) such q that $e(x^q) = \min_{\bar{q}} e(x^{\bar{q}})$ or $b_q = 0$, respectively] the inequality

$$E(\mathbb{w}) \geqslant \beta(E_W(\mathbb{w}) - E_{\pi(W)}(\mathbb{I}_0) - \min_q e(x^q)[|W| - |\pi(W)|]) - 3^\nu |W| - ||W| - |\pi(W)||$$

Obviously, $\min_q e(x^q)$ can be supplied by $e(x^1)$ or $e(x^2)$ in the last inequality. Now the "Peierls condition" (WP) can be applied and one gets

$$E(\mathbb{w}) \geqslant (\beta\rho_2 - 3^\nu)\,|W| - ||W| - |\pi(W)||$$

Using the expression for $\tilde{Z}(\mathbb{I}, U \mid y)$ from Lemma 4.4, we obtain the inequality

$$\tilde{Z}(\mathbb{I}, U \mid y) \leqslant \exp\left\{-\sum_{\mathbb{w} \in \mathbb{W}(\mathbb{I})} [(\beta\rho_2 - 3^\nu)\,|W| - ||W| - |\pi(W)||]\right\} \times \exp(4\,|I \cap U_R|)$$

with the help of the estimate (B11) again. We may write

$$\sum_{\mathbb{w} \in \mathbb{W}(\mathbb{I})} [|W| - |\pi(W)|] + |I_0 \cap U_R|$$

instead of $|I \cap U_R|$ and conclude that

$$\tilde{Z}(\mathbb{I}, U \mid y) \leqslant \exp\left\{-\sum_{\mathbb{w} \in \mathbb{W}(\mathbb{I})} [(\beta\rho_2 - 3^\nu)|W| - 5\,|W|]\right\} \exp(4\,|I_0 \cap U_R|) \equiv K_B(I)$$

because

$$4(|W| - |\pi(W)|) + ||W| - |\pi(W)|| \leqslant 5\,|W|$$

Obviously $K_B(I)$ depends on B, I_0, and R, but actually not on U. The sum can be bounded in the following way:

$$\sum_{\mathbb{I} \in \mathscr{I}(V)} K_B(I) \leqslant \exp(4\,|I_0 \cap V_R|) \prod_{i \in I_0 \cap V_R} \sum_{\substack{\mathbb{w} \in \mathbb{W} \\ i \in W}} \exp[-(\beta\rho_2 - 3^\nu - 5)\,|W|]$$

The support of walls are connected sets and therefore we can use the estimate already used for contours:

$$|\{\mathbb{w} \mid W \ni i,\ |W| = n\}| \leqslant c^n$$

Hence

$$\sum_{\substack{\mathbb{W}\in\mathbb{W}\\ i\in W}} \exp[-(\beta\rho_2-3^\nu-5)|W|]$$

$$\leqslant \sum_{n=R^\nu}^{\infty} c^n \exp[-(\beta\rho_2-3^\nu-5)\,n]$$

$$\leqslant \frac{\{c\exp[-(\beta\rho_2-3^\nu-5)]\}^{R^\nu}}{1-c\exp[-(\beta\rho_2-3^\nu-5)]}\leqslant 1$$

if

$$\beta\rho_2 \geqslant \log(2c)+3^\nu+5$$

We introduced the concepts of walls, standard walls, and admissible families of walls in Definitions 2.3 and 2.4. Let us introduce the notation $\mathbb{W}(V)$ for the set of all standard walls with supports contained in V_R, and the notation $\mathscr{W}^{\mathrm{a}}(V)$ for the set of all admissible families $\mathbb{V}\subset\mathbb{W}(V)$ of standard walls.

We shall use Lemma 2.2 with the following trivial supplement.

Lemma 4.6. The mapping $W(\cdot)$: $\mathscr{I}\to\mathscr{W}^{\mathrm{co}}$ from Lemma 2.2 satisfies the equality $W(\mathscr{I}(V))=\mathscr{W}^{\mathrm{a}}(V)$.

Our next step will be to rewrite $\tilde{Z}(\mathbb{I}, V\,|\, y)$ as a sum over triplets $\mathbb{T}=(\mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2)\in\mathscr{T}(V)$ defined so that $\mathbb{T}_0\in\mathscr{W}^{\mathrm{a}(\cdot,f)}(V)$, $\mathbb{T}_1, \mathbb{T}_2$ are finite subsets of $\mathscr{K}_1^{\mathrm{cl}(\cdot,f)}$ or $\mathscr{K}_2^{\mathrm{cl}(\cdot,f)}$, respectively, and the supports of elements of $\mathbb{T}_q$, $q=1, 2$, intersect V_R and $I(\mathbb{T}_0)$, i.e., the suport of the only interface $\mathbb{I}(\mathbb{T}_0)$ determined by $\mathbb{T}_0$. Let us use $\mathscr{T}$ instead of $\mathscr{T}(\mathbb{Z}^\nu)$, and let us agree to use $\mathbb{T}_q$, $q=0, 1, 2$, in the above meaning whenever $\mathbb{T}\in\mathscr{T}$. The following lemma yields a base for rewriting $\tilde{Z}(V\,|\, y)$ in a form similar to the partition of some contour model.

Lemma 4.7. Under the assumptions of Lemma 4.5, one gets for $\mathbb{I}\in\mathscr{I}(V)$ that

$$\tilde{Z}(\mathbb{I}, V\,|\, y)=\prod_{\mathbb{w}\in\mathbb{T}_0=\mathbb{W}(\mathbb{I})} e^{-E(\mathbb{w})}\sum_{\mathbb{T}\in\mathscr{T}(V),\,\mathbb{I}(\mathbb{T}_0)=\mathbb{I}}\ \prod_{q=1}^{2}\ \prod_{\mathbb{C}\in\mathbb{T}_q} f^q_{V,I}(\mathbb{C})$$

where

$$f^q_{V,I}(\mathbb{C})=\exp\left\{-\Phi^T_q(\mathbb{C})\left[\frac{|C\cap V_q(I)|}{|C|}-\chi_q(C)\frac{|C\cap V_R|}{|C|}\right]\right\}-1$$

whenever $\mathbb{C}\in\mathscr{K}^{\mathrm{cl}}_q$.

Proof. We use Lemma 4.5(b). According to Lemma 4.6, we know that $\mathbb{W}(\mathscr{I}(V)) = \mathbb{W}(V)$. Therefore, it remains to prove the equality

$$\exp\left\{-\sum_{q=1}^{2}\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\text{cl}}\\ C\cap I\neq\varnothing}}\Phi_q^T(\mathbb{C})\left[\frac{|C\cap V_q(I)|}{|C|}-\chi_q(C)\frac{|C\cap V_R|}{|C|}\right]\right\}$$

$$=\sum_{(\mathbb{T}_0=\mathbb{W}(I),\,\mathbb{T}_1,\,\mathbb{T}_2)\in\mathscr{T}(V)}\prod_{q=1}^{2}\prod_{\mathbb{C}\in\mathbb{T}_q} f_{V,I}^q(\mathbb{C})$$

This equality follows from the observation that

$$\exp\left(\sum_{n\in N} a_n\right)=\prod_{n\in N}[(\exp a_n-1)+1]=\sum_{K\subset N\text{ finite}}\prod_{n\in K}(\exp a_n-1)$$

for countable N if $\sum |a_n| < \infty$. The inequality $\sum_{q,\mathbb{C}} |\Phi_q^T(\mathbb{C})|\,|[\cdots]| < 4$ follows from (B4′).

The set

$$\left(\bigcup_{\mathbb{W}\in\mathbb{T}_0} W\cup\bigcup_{\mathbb{C}\in\mathbb{T}_1} C\cup\bigcup_{\mathbb{C}\in\mathbb{T}_2} C\right)$$

is called the support of $\mathbb{T} = (\mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2) \in \mathscr{T}$ and is denoted by $\text{supp}(\mathbb{T})$ or simply by T.

Let $\mathbb{T} \in \mathscr{T}(V)$ and $\mathbb{a} = (\mathbb{a}_0, \mathbb{a}_1, \mathbb{a}_2)$, where $\mathbb{a}_0 \subset \mathbb{T}_0$, $\mathbb{a}_1 \subset \mathbb{T}_1$, $\mathbb{a}_2 \subset \mathbb{T}_2$ are such that

$$\pi(\text{supp}(\mathbb{a}))\equiv\left(\bigcup_{\mathbb{W}\in\mathbb{a}_0} W\cup\bigcup_{\mathbb{C}\in\mathbb{a}_1} C\cup\bigcup_{\mathbb{C}\in\mathbb{a}_2} C\right)$$

is a connected component of $\pi(\text{supp}(\mathbb{T}))$. Then we say that $\mathbb{a}$ is an aggregate of $\mathbb{T}$. The triplet $(\mathbb{a}_0, \mathbb{a}_1, \mathbb{a}_2)$ is called an aggregate (in V) if it is an aggregate of some triplet from $\mathscr{T}$ ($\mathscr{T}(V)$). We again agree that $\mathbb{a}_0$, $\mathbb{a}_1$, $\mathbb{a}_2$ have the above meaning whenever $\mathbb{a}$ is an aggregate and we denote the support of $\mathbb{a}$ by A.

If $\mathbb{a}$ is the only agrregate of a triplet, $\mathbb{a}$ is called a standard aggregate. The set of all standard aggregates from $\mathscr{T}(V)$ is denoted by $\mathbb{A}(V)$. We use the notation $\mathscr{A}(V)$ for the set of finite subsets of $\mathbb{A}(V)$ consisting of standard aggregates such that for any two of them, say $\mathbb{a}$, $\bar{\mathbb{a}}$, the set $\pi(\text{supp}(\mathbb{a})) \cup \pi(\text{supp})\bar{\mathbb{a}}))$ is disconnected.

The proof of the following geometrical assertion is scetched in Appendix A.

Lemma 4.8. (a) For any aggregate $\mathbb{a} = (\mathbb{a}_1, \mathbb{a}_1, \mathbb{a}_2)$ of $\mathbb{T} \in \mathscr{T}(V)$ there is one and only one $h = h(\mathbb{a}) \in \mathbb{Z}$ such that the shift

$T_h\mathbb{a} = (\{T_h\mathbb{w} \mid \mathbb{w} \in \mathbb{a}_0\}, \{T_h\mathbb{C} \mid \mathbb{C} \in \mathbb{a}_1\}, \{T_h\mathbb{C} \mid \mathbb{C} \in \mathbb{a}_2\})$ is in $\mathbb{A}(V)$. The shift $T_h\mathbb{a}$ is called $\mathbb{a}$ in the standard position.

(b) The mapping that ascribes to a triplet $\mathbb{T} \in \mathscr{T}(V)$ its aggregates in standard positions is a one-to-one mapping $(\mathbb{A}(\cdot))$ from $\mathscr{T}(V)$ onto $\mathscr{A}(V)$.

Finally, we express $\tilde{Z}(\mathbb{I}, V \mid y)$ in terms of certain contour model in the sense of the abstract definition from Appendix B. The assertions of the following lemma are immediate consequences of Lemmas 4.7 and 4.8.

Lemma 4.9. Let us denote

$$\Psi^V(\mathbb{a}) = \prod_{\mathbb{w} \in \mathbb{a}_0} e^{-E(\mathbb{w})} \prod_{q=1}^{2} \prod_{\mathbb{C} \in \mathbb{a}_q} f^q_{V, I(\mathbb{a}_0)}(\mathbb{C})$$

whenever $\mathbb{a} \in \mathbb{A}(V)$ is a standard aggregate, and recall that V is a cylinder set with a finite base. Then, under the assumptions of Lemma 4.5, one has:

(a)
$$\tilde{Z}(V \mid y; \beta H) = \mathscr{Z}(\mathbb{A}(V); \Psi^V)$$

where

$$\tilde{Z}(V \mid y; \beta H) = \frac{Z(V \mid y; \beta H)}{N(V \mid y)}$$

[see Lemma 4.3 for the definition of $N(V \mid y)$] and

$$\mathscr{Z}(\mathbb{A}(V); \Psi^V) = \sum_{\mathbb{R} \in \mathscr{A}(V)} \prod_{\mathbb{a} \in \mathbb{R}} \Psi^V(\mathbb{a})$$

(cf. Appendix B).

(b)
$$P^{\mathscr{I}}_V(\mathbb{I}(\mathbb{V})) = \sum_{\substack{\mathbb{S} \in \mathscr{A}(V) \\ \mathbb{W}(\mathbb{S}) = \mathbb{V}}} \rho_{\mathbb{A}(V)}(\mathbb{S}; \Psi^V)$$

where $P^{\mathscr{I}}_V$ is the probability defined in Lemma 4.5 and

$$\rho_{\mathbb{A}(V)}(\mathbb{S}; \Psi^V) = \left[\prod_{\mathbb{s} \in \mathbb{S}} \Psi^V(\mathbb{s})\right] \Big/ \mathscr{Z}(\mathbb{A}(V); \Psi^V)$$

for $\mathbb{S} \in \mathscr{A}(V)$ (cf. Appendix B).

When proving Lemma 4.9 from Lemmas 4.7 and 4.8, we use the observation that if $\mathbb{T} \in \mathscr{T}(V)$ and $\mathbb{a}$ is an aggregate of $\mathbb{T}$ in standard position $[\in \mathbb{A}(\mathbb{T})]$, then $f^q_{V, I(a_0)}(\mathbb{C}) = f^q_{V, I(T_0)}(\tilde{\mathbb{C}})$ for $\mathbb{C} \in a_q$, $q = 1, 2$, and $\tilde{\mathbb{C}} = T_h\mathbb{C}$ for the only $h \in \mathbb{Z}$ such that $T_h\mathbb{a}$ is an aggregate of $\mathbb{T}$. We use this fact without further comment in subsequent sections.

5. STUDY OF THE AGGREGATE CONTOUR MODEL

In Lemma 4.9 we rewrote both $\tilde{Z}(\mathbb{I}, V \mid y)$ and $P_V^{\mathscr{I}}(\mathbb{I})$ in terms of a contour model with aggregates playing the role of contours and with the contour functional Ψ^V for the cylindrical volume V with finite base. However, these functionals depend on V (for aggregates touching V^c) and this could cause some trouble when studying the limit over V's. One observes easily that the definition of $f^q_{V,I}$, and thus also of Ψ^V, can be directly transfered to the case of arbitrary cylindrical volume V. Therefore, we may and shall use Ψ^V for any cylindrical subset V of $\mathbb{Z}^\nu$, for example, $\mathbb{Z}^\nu$ itself. In Lemma 5.2 we show that the assumptions of the functionals Ψ^V needed for an application of Theorem B.1 are fulfilled. Moreover, it will turn out that the corresponding inequalities are independent of V. Proposition 5.3 is a direct application of Theorem B.1.

5.1. Contour Functionals Ψ^V

Let us recall that, given a cylindrical volume V in $\mathbb{Z}^\nu$, the functional Ψ^V is defined by the equality ($\mathbb{a} \in \mathbb{A}$)

$$\Psi^V(\mathbb{a}) = \exp\left\{-\sum_{\mathbb{w} \in \mathbb{a}_0} E(\mathbb{w}) \prod_{q=1}^{2} \prod_{\mathbb{C} \in \mathbb{a}_q} f^q_{V, I(\mathbb{a}_0)}(\mathbb{C})\right\}$$

where

$$f^q_{V,I}(\mathbb{C}) = \exp\left\{-\Phi_q^T(\mathbb{C})\left[\frac{|C \cap V_q(I)|}{|C|} - \chi_q(C)\frac{|C \cap V_R|}{|C|}\right]\right\} - 1$$

where $\mathbb{C} \in \mathscr{K}_q^{\mathrm{cl}}$ and I is the support of some interface $\mathbb{I} \in \mathscr{I}(V)$.

The assumptions of Theorem B.1 for the functionals Ψ^V are verified in Lemma 5.2. For its proof we need the following estimate.

Lemma 5.1. Let $\beta\rho_1 \geqslant c_3$. Then

$$\sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\mathrm{cl}} \\ i \in C}} |f^q_{V,I}(\mathbb{C})| \exp[\omega \|\mathbb{C}\|] \leqslant \kappa$$

whenever

$$\omega \leqslant \beta\rho_1 - c_3 - \frac{2 + \log 2}{(2R+1)^\nu} + \frac{\log \kappa}{(2R+1)^\nu}$$

and $\kappa \leqslant 2e^2$, $q = 1$ or 2, V is a cylindrical volume in $\mathbb{Z}^\nu$, I is the support of some interface $\mathbb{I} \in \mathscr{I}(V)$, and $i \in \mathbb{Z}^\nu$.

Proof. Taking into account that

$$\sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\text{cl}} \\ C \ni i}} \Phi_q^T(\mathbb{C}) \exp(\tilde{\omega} \|\mathbb{C}\|) \leqslant 1$$

for $\tilde{\omega} = \beta\rho_1 - c_3$ according to Proposition 3.2 and Theorem B.2, that $|e^u - 1| \leqslant e^v |u|$ if $|u| \leqslant v$, and that the support of any cluster $\mathbb{C}$ contains at least $(2R+1)^\nu$ lattice sites, we have

$$\begin{aligned}
&\sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\text{cl}} \\ C \ni i}} |f_{V,I}^q(\mathbb{C})| \exp(\omega \|\mathbb{C}\|) \\
&\quad = \sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\text{cl}} \\ C \ni i}} \left| \exp\left\{ -\Phi_q^T(\mathbb{C}) \left[\frac{|C \cap V_q(I)|}{|C|} - \chi_q(C) \frac{|C \cap V_R|}{|C|} \right] \right\} - 1 \right| \\
&\quad\quad \times \exp(\omega \|\mathbb{C}\|) \\
&\quad \leqslant \sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\text{cl}} \\ C \ni i}} 2e^2 |\Phi_q^T(\mathbb{C})| \exp(\omega \|C\|) \\
&\quad \leqslant 2e^2 \exp[(\omega - \tilde{\omega})(2R+1)^\nu] \leqslant \kappa
\end{aligned}$$

Lemma 5.2. Recalling that we introduced the "thickness" t by $t = |\pi\{i\}|/2R$ for any $i \in \mathbb{Z}^\nu$, let us define

$$\xi = \log(4c) + t + \frac{\log[3c(\nu - 1)]}{2R}$$

Let $\beta\rho_1 \geqslant c_3$. Then

$$\sum_{\substack{\mathbb{a} \in \mathbb{A}(V) \\ \pi(A) \ni i}} \exp(|\pi(A)| + \omega \|\mathbb{a}\|) \Psi^V(\mathbb{a}) \leqslant 1$$

with

$$\|\mathbb{a}\| = \sum_{\mathbb{w} \in \mathbb{a}_0} |W| + \sum_{\mathbb{C} \in \mathbb{a}_1 \cup \mathbb{a}_2} \|\mathbb{C}\|$$

whenever

$$\omega \leqslant \min\left(\beta\rho_1 - c_3 - \frac{2 + \log 4}{(2R+1)^\nu} - t\xi, \beta\rho_2 - 3^\nu - t - \xi \right)$$

Proof. Let $\mathscr{P}$ denote the set of finite, connected subsets $P \subset I_0$ such that $\pi(P) = P$ and, if $\mathbb{a}_0$ is an admissible family of walls in V_R and $P \in \mathscr{P}$, let $I(\mathbb{a}_0, P) = I(\mathbb{a}_0) \cap \pi^{-1}(P)$. Then

$$\sum_{\substack{\mathbb{a} \in \mathbb{A}(V) \\ \pi(A) \ni i}} \exp[|\pi(A)| + \omega \|\mathbb{a}\|] \, \Psi^V(\mathbb{a})$$
$$\leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{\substack{\mathbb{a} \in \mathbb{A}(V) \\ \pi(A) = P}} \exp(\omega \|\mathbb{a}\|) \exp\left\{- \sum_{\mathbb{w} \in \mathbb{a}_0} E(\mathbb{w})\right\}$$
$$\times \prod_{q=1}^{2} \prod_{\mathbb{C} \in \mathbb{a}_q} f^q_{V, I(\mathbb{a}_0)}(\mathbb{C}) = (1)$$

Using the inequality $E(\mathbb{w}) \geqslant (\beta\rho_2 - 3^\nu) \, |W| - | \, |W| - |\pi(W)| \, |$ derived in the proof of Lemma 4.5, we have $E(\mathbb{w}) \geqslant (\beta\rho_2 - 3^\nu - t) \, |W|$, since $| \, |W| - |\pi(W)| \, | \leqslant t \, |W|$. Hence

$$(1) \leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{\substack{\pi(\operatorname{supp} \mathbb{a}_0) \subset P \\ \mathbb{a}_0 \in \mathscr{W}^{\mathrm{a}}(V)}} \exp[(\omega - \beta\rho_2 + 3\nu + t) \, \|\mathbb{a}_0\|]$$
$$\times \sum_{(\mathbb{a}_1, \mathbb{a}_2) \in \mathbb{B}(\mathbb{a}_0, P)} \prod_{q=1}^{2} \exp(\omega \, \|\mathbb{a}_q\|) \prod_{\mathbb{C} \in \mathbb{a}_q} |f^q_{V, I(\mathbb{a}_0)}(\mathbb{C})|$$
$$\leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{\substack{\mathbb{a}_0 \in \mathscr{W}^{\mathrm{a}}(V) \\ \pi(\operatorname{supp} \mathbb{a}_0) \subset P}} \exp[(\omega - \beta\rho_2 + 3^\nu - t - \xi) \, \|\mathbb{a}_0\|$$
$$- \xi \, |I(\mathbb{a}_0, P)| \,]$$
$$\times \sum_{(\mathbb{a}_1, \mathbb{a}_2) \in \mathbb{B}(\mathbb{a}_0, P)} \prod_{q=1}^{2} \exp[(\omega + t\xi) \, \|\mathbb{a}_q\|] \prod_{\mathbb{C} \in \mathbb{a}_q} |f^q_{V, I(\mathbb{a}_0)}(\mathbb{C})|$$
$$= (2)$$

We use the notation $\mathbb{B}(\mathbb{a}_0, P)$ for the set $\{(\mathbb{a}_1, \mathbb{a}_2) | (\mathbb{a}_0, \mathbb{a}_1, \mathbb{a}_2) \in \mathbb{A}(\mathbb{Z}^\nu),$ $\pi(A) = P\}$, and in the last inequality we used the fact that $\|\mathbb{a}_0\| + t \, \|\mathbb{a}_1\| + t \, \|\mathbb{a}_2\| \geqslant |I(\mathbb{a}_0, P)|$ whenever $\pi(A) \supset P$. According to Lemma 5.1 [using the inequality $\omega + t\xi \leqslant \beta\rho_1 - c_3 - (2 + \log 4)/(2R+1)^\nu$] and the inequality $\omega - \beta\rho_2 + 3^\nu + t + \xi \leqslant 0$, one has

$$(2) \leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{\substack{\mathbb{a}_0 \in \mathscr{W}^{\mathrm{a}}(V) \\ \pi(\operatorname{supp} \mathbb{a}_0) \subset P}} \exp[-\xi \, |I(\mathbb{a}_0, P)| \,]$$
$$\times \prod_{q=1}^{2} \prod_{j \in I(\mathbb{a}_0, P)} \sum_{k=0}^{\infty} \left(\sum_{\substack{\mathbb{C} \in \mathscr{K}^{\mathrm{cl}}_q \\ C \ni j}} \{\exp[(\omega + t\xi) \, \|\mathbb{C}\|]\} \, |f^q_{V, I(\mathbb{a}_0)}(\mathbb{C})| \right)^k$$
$$\leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{\substack{\mathbb{a}_0 \in \mathscr{W}^{\mathrm{a}}(V) \\ \pi(\operatorname{supp} \mathbb{a}_0) \subset P}} \exp[-(\xi - \log 4) \, |I(\mathbb{a}_0, P)| \,]$$
$$\leqslant \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp |P|) \sum_{n = |P|/t}^{\infty} \exp[-(\xi - \log 4) n] \, c^n = (3)$$

where we used the inequality $|I(\mathbb{a}_0, P)| \geqslant |P|/t$ and the fact that the number of admissible families $\mathbb{a}_0$ of walls such that $I(\mathbb{a}_0, P)$ is conected, it contains a particular site in $\partial(\pi^{-1}(P)) \cap I_0$, and $|I(\mathbb{a}_0, P)| = n$, is bounded by c^n. [This is the same estimate as (B.10) for the number of contours.]

Since $\xi > \log(4c)$, we have

$$(3) = \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} (\exp|P|) \frac{\exp[(\log(4c) - \xi)|P|/t]}{1 - \exp[\log(4c) - \xi]}$$

$$\leqslant 2 \sum_{\substack{P \in \mathscr{P} \\ P \ni i}} \exp[(t + \log(4c) - \xi)|P|/t]$$

$$\leqslant 2 \sum_{n=1}^{\infty} [c(v-1)]^n \exp[(t + \log(4c) - \xi)\, 2Rn] = (4)$$

In the last inequality we used the estimate

$$|\{P \in \mathscr{P} \mid P \ni i, |P| = 2Rtn\}| \leqslant [c(v-1)]^n$$

(cf. B.10). Since

$$\xi \geqslant \frac{\log[3c(v-1)]}{2R} + t + \log(4c)$$

we have

$$(4) = 2\frac{\exp\{2R[t + \log(4c) - \xi] + \log c(v-1)\}}{1 - \exp\{2R[t + \log(4c) - \xi] + \log c(v-1)\}} \leqslant 1$$

Our last task in this section is to use Lemma 5.2 to get a version of Theorem B.1 with aggregates playing the role of contours. We shall say that aggregates $\mathbb{a}_1$ and $\mathbb{a}_2$ are incompatible iff $d(\pi(\operatorname{supp} \mathbb{a}_1), \pi(\operatorname{supp} \mathbb{a}_2)) \leqslant 1$. Then we can use the notation from Appendix B with $\mathbb{K}$ and $\mathscr{K}$ replaced by $\mathbb{A}$ and $\mathscr{A}$; thus, e.g., $\mathscr{A}^{\mathrm{cl}}$ is the set of clusters of aggregates. Let us take $\omega(\mathbb{a}) = |\pi(A)|$; $l(\mathbb{a}) = \|\mathbb{a}\|$ for $\mathbb{a} \in \mathbb{A}$, $\|\mathbb{C}\| = \sum_{\mathbb{a} \in \mathbb{C}} \|\mathbb{a}\|$ for $\mathbb{C} \in \mathscr{A}^{\mathrm{cl}}$, and denote $\operatorname{supp} \mathbb{S} = \bigcup_{\mathbb{a} \in \mathbb{S}} \operatorname{supp} \mathbb{a}$ whenever $\mathbb{S} \subset \mathbb{A}$. Finally let us introduce the constants

$$c_5 = c_5(v, |S|, R, t) = c_3 + t\xi + \frac{2 + \log 4}{(2R+1)^v}$$

and

$$c_6 = c_6(v, |S|, R, t) = 3^v + t + \xi$$

with ξ defined in Lemma 5.2 above. Using this lemma and Theorem B.1 together with the Remark following it, we get the following result:

Proposition 5.3. Let $\beta\rho_1 \geqslant c_5$ and $\beta\rho_2 \geqslant c_6$ and let V be a cylinder set $\mathbb{Z}^\nu$. Then there exists a unique function $\Psi^{V,T}$: $\mathcal{C}^{\text{cl}} \to \mathbb{R}$ such that

$$\log \mathscr{Z}(\mathbb{B}; \Psi^V) = \sum_{\mathbb{C} \in \mathcal{C}^{\text{cl}}(\mathbb{B})} \Psi^{V,T}(\mathbb{C})$$

for every $\mathbb{B} \subset \mathbb{A}(U)$, where U is a cylinder set with a finite base, and that for each $i \in I_0$ and $\omega \leqslant \min(\beta\rho_1 - c_5, \beta\rho_2 - c_6)$, one has

$$\sum_{\substack{\pi(C) \ni i \\ \mathbb{C} \in \mathcal{C}^{\text{cl}}}} |\Psi^{V,T}(\mathbb{C})| \, e^{\omega \|\mathbb{C}\|} \leqslant 1$$

Moreover,

$$\Psi^{V,T}(\mathbb{C}) = \sum_{\mathbb{D} \subset \mathbb{C}} (-1)^{|\mathbb{C}| - |\mathbb{D}|} \log \mathscr{Z}(\mathbb{D}; \Psi^V)$$

for every $\mathbb{C} \in \mathcal{C}^{\text{cl}}$.

For every $\mathbb{S} \in \mathcal{C}^{f,\text{co}}$ there exists a unique function $\Delta^V_{\mathbb{S}}$: $\mathcal{C}^f \to \mathbb{C}$ such that

$$\rho_{\mathbb{B}}(\mathbb{S}; \Psi) = \sum_{\mathbb{C} \in \mathcal{C}^{\text{cl}}(\mathbb{B})} \Delta^V_{\mathbb{S}}(\mathbb{C})$$

for every $\mathbb{B} \subset \mathbb{A}$ and that

$$\sum_{\mathbb{C} \supset \mathbb{S}} |\Delta^V_{\mathbb{S}}(\mathbb{C})| \, e^{\omega \|\mathbb{C}\|} \leqslant e^{|\pi(\text{supp } \mathbb{S})|} \left| \prod_{\mathbb{a} \in \mathbb{S}} \Psi^V(\mathbb{a}) \right|$$

The function $\Delta^V_{\mathbb{S}}$ is given by

$$\Delta^V_{\mathbb{S}}(\mathbb{C}) = \sum_{\mathbb{D} \subset \mathbb{C}} (-1)^{|\mathbb{C}| - |\mathbb{D}|} \rho_{\mathbb{D}}(\mathbb{S}; \Psi^V)$$

and

$$\Delta^V_{\mathbb{S}_1 \cup \mathbb{S}_2}(\mathbb{C}_1 \cup \mathbb{C}_2) = \Delta^V_{\mathbb{S}_1}(\mathbb{C}_1) \, \Delta^V_{\mathbb{S}_2}(\mathbb{C}_2)$$

whenever $\mathbb{a}_1 \in \mathbb{S}_1 \cup \mathbb{C}_1$ is compatible with every $\mathbb{a}_2 \in \mathbb{S}_2 \cup \mathbb{C}_2$.

6. PROOF OF THEOREM 2 AND THEOREM 1

6.1. The estimates of Section 5 show that our aggregate model is a "well-behaving" model, exhibiting all the nice features of the contour models satisfying the condition (B.4), in particular the exponential decay of correlations.

Our primary task is, however, a control over the behavior of the walls of the original model. Thus, we have to "extract" relevant information for walls from our information about the aggregate model. This is the aim of this section and in pursuing it we prove here Theorems 1 and 2:

$$P_U^{\mathscr{I}}(\mathbb{I}) = \sum_{\substack{\mathbb{S} \in \mathfrak{A}^{\mathrm{co}}(U) \\ \mathbb{W}(\mathbb{S}) = \mathbb{W}(\mathbb{I})}} \rho_{\mathbb{A}(U)}(\mathbb{S}; \Psi^U) \tag{6.1}$$

6.1.1. We now prove that the correlations defined for $\mathbb{V} \in \mathscr{W}^{\mathrm{co},f}(U)$ by

$$\rho_U^{\mathscr{I}}(\mathbb{V}) = P_U^{\mathscr{I}}(\{\mathbb{I} \in \mathscr{I}(U) \mid \mathbb{W}(\mathbb{I}) \supset \mathbb{V}\})$$

have the properties claimed by (a)–(c) in Theorem 2(i). The proof is based on (6.1) and Proposition 5.3.

We shall use repeatedly the following estimate, which is a corollary to Proposition 5.3:

Lemma 6.1. Let $\beta\rho_1 \geqslant c_5$, $\beta\rho_2 \geqslant c_6$, $0 \leqslant \omega \leqslant \min(\beta\rho_1 - c_5, \beta\rho_2 - c_6)$, and let $\mathscr{S} \subset \mathfrak{A}^{\mathrm{co}}$, $\mathscr{C} \subset \mathfrak{A}^{\mathrm{cl}}$, and a finite $M \subset \mathbb{Z}^\nu$ be given such that $\pi(M) = M$ and $\mathbb{a} \in \mathbb{S}$ implies $\pi(A) \cap M \neq \varnothing$ whenever $\mathbb{S} \in \mathscr{S}$. Then

$$\sum_{\mathbb{S} \in \mathscr{S}} \sum_{\mathbb{C} \in \mathscr{C}} |\Delta_{\mathbb{S}}^U(\mathbb{C})| \leqslant 2^{|M|} \exp[-\omega(\inf_{\mathscr{C}} \|\mathbb{C}\| + \inf_{\mathscr{S}} \|\mathbb{S}\|)]$$

Proof.

$$\begin{aligned}
&\sum\sum |\Delta_{\mathbb{S}}^U(\mathbb{C})| \\
&\quad \leqslant \sum\sum [\exp(-\omega \inf_{\mathscr{C}} \|\mathbb{C}\|) \exp(\omega \|\mathbb{C}\|)]\, |\Delta_{\mathbb{S}}^U(\mathbb{C})| \\
&\quad \leqslant [\exp(-\omega \inf_{\mathscr{C}} \|\mathbb{C}\|)] \left\{ \sum_{\mathbb{S}} [\exp \pi(\operatorname{supp} \mathbb{S})] \prod_{\mathbb{a} \in \mathbb{S}} |\Psi^U(\mathbb{a})| \right\} \\
&\quad \leqslant \left\{ \sum_{\mathbb{S}} [\exp \pi(\operatorname{supp} \mathbb{S}) \exp(\omega \|\mathbb{S}\|)] \prod_{\mathbb{a} \in \mathbb{S}} |\Psi^U(\mathbb{a})| \right\} \\
&\qquad \times \exp[-\omega(\inf_{\mathscr{C}} \|\mathbb{C}\| + \inf_{\mathscr{S}} \|\mathbb{S}\|)] \\
&\quad \leqslant \prod_{i \in M} \left\{ 1 + \sum_{\mathbb{a}: i \in \pi(A)} \exp[\pi(A) + \omega \|\mathbb{a}\|]\, |\Psi^U(\mathbb{a})| \right\} \\
&\qquad \times \exp[-\omega(\inf \|\mathbb{C}\| + \inf \|\mathbb{S}\|)] \\
&\quad \leqslant 2^{|M|} \exp[-\omega(\inf_{\mathscr{C}} \|\mathbb{C}\| + \inf_{\mathscr{S}} \|\mathbb{S}\|)]
\end{aligned}$$

Proof of Theorem 2(i)(a) for a Cylinder U with a Finite Base. Let us denote the set

$$\{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U)\mid \mathbb{W}(\mathbb{S})\supset\mathbb{V},\ \mathbb{a}\in\mathbb{S}\Rightarrow\mathbb{V}\cap\mathbb{W}(\mathbb{a})\neq\varnothing\}$$

by $\mathcal{A}^{\mathrm{co}}(U, V)$. Then, due to Lemma 4.9,

$$\rho_U^{\mathscr{I}}(\mathbb{V})=\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U,\mathbb{V})}\rho_{\mathbb{A}(U)}(\mathbb{S};\Psi^U)$$

Proposition 5.3 and Lemma 6.1 imply that

$$\rho_U^{\mathscr{I}}(\mathbb{V})=\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U,\mathbb{V})}\ \sum_{\substack{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}\\ \mathbb{C}\subset\mathbb{A}(U)}}\Delta_{\mathbb{S}}^{U}(\mathbb{C})\leqslant\exp[-(2\omega-t\log 2)\,\|\mathbb{V}\|]$$

because $\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U,\mathbb{V})$ implies $[\mathbb{a}\in\mathbb{S}\Rightarrow\pi(A)\cap\pi\ (\mathrm{supp}\ \mathbb{V})\neq\varnothing]$, $\Delta_{\mathbb{S}}^{U}(\mathbb{C})\neq 0\Rightarrow\mathbb{C}\supset\mathbb{S}$, $\|\mathbb{S}\|\geqslant\|\mathbb{V}\|$ for $\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U, \mathrm{V})$, and $|\pi(\mathrm{supp}\ \mathbb{V})|\leqslant t\,\|\mathbb{V}\|$.

Proof of Theorem 2(i)(b) for a Cylinder U with a Finite Base. First we express the difference $\rho_{U_1}^{\mathscr{I}}(\mathbb{V})-\rho_{U_2}^{\mathscr{I}}(\mathbb{V})$ using the notation $\mathcal{A}^{\mathrm{co}}(U,\mathbb{V})$ introduced in the preceding paragraph.

$$\begin{aligned}\rho_{U_1}^{\mathscr{I}}(\mathbb{V})-\rho_{U_2}^{\mathscr{I}}(\mathbb{V})&=\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_1,\mathbb{V})}\rho_{\mathbb{A}(U_1)}(\mathbb{S};\Psi^{U_1})\\&\quad-\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_2,\mathbb{V})}\rho_{\mathbb{A}(U_2)}(\mathbb{S};\Psi^{U_2})\\&=\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_1,\mathbb{V})}\ \sum_{\substack{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_1)\\ (\mathbb{C}\supset\mathbb{S})}}\Delta_{\mathbb{S}}^{U_1}(\mathbb{C})\\&\quad-\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_2,\mathbb{V})}\ \sum_{\substack{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_2)\\ (\mathbb{C}\supset\mathbb{S})}}\Delta_{\mathbb{S}}^{U_2}(\mathbb{C})\end{aligned}$$

Since $\Delta_{\mathbb{S}}^{U_1}(\mathbb{C})=\Delta_{\mathbb{S}}^{U_2}(\mathbb{C})$ for $\mathrm{supp}\,\mathbb{C}\subset U_1\cap U_2$, $\mathrm{supp}\,\mathbb{S}\subset U_1\cap U_2$, according to Proposition 5.3, we get

$$\begin{aligned}\rho_{U_1}^{\mathscr{I}}(\mathbb{V})-\rho_{U_2}^{\mathscr{I}}(\mathbb{V})&=\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_1\cap U_2,\mathbb{V})}\ \sum_{\substack{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_1)\\ \mathbb{C}\notin\mathcal{A}^{\mathrm{cl}}(U_2)}}\Delta_{\mathbb{S}}^{U_1}(\mathbb{C})\\&\quad-\sum_{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_1\cap U_2,\mathbb{V})}\ \sum_{\substack{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_2)\\ \mathbb{C}\notin\mathcal{A}^{\mathrm{cl}}(U_1)}}\Delta_{\mathbb{S}}^{U_2}(\mathbb{C})\\&\quad+\sum_{\substack{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_1,\mathbb{V})\\ \mathbb{S}\notin\mathcal{A}^{\mathrm{co}}(U_2,\mathbb{V})}}\ \sum_{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_1)}\Delta_{\mathbb{S}}^{U_1}(\mathbb{C})\\&\quad-\sum_{\substack{\mathbb{S}\in\mathcal{A}^{\mathrm{co}}(U_2,\mathbb{V})\\ \mathbb{S}\notin\mathcal{A}^{\mathrm{co}}(U_1,\mathbb{V})}}\ \sum_{\mathbb{C}\in\mathcal{A}^{\mathrm{cl}}(U_2)}\Delta_{\mathbb{S}}^{U_2}(\mathbb{C})\end{aligned}$$

We can estimate each of these four sums using Lemma 6.1 as in the proof of (a) to get

$$|\rho_{U_1}^{\mathscr{I}}(\mathbb{V})-\rho_{U_2}^{\mathscr{I}}(\mathbb{V})| \leqslant 4\exp[-(\omega-t\log 2)\|\mathbb{V}\|-\omega\, d(\operatorname{supp}\mathbb{V}, U_1 \div U_2)]$$

Proof of Theorem 2(i)(c) for a Cylinder U with a Finite Base. The difference $\rho_U^{\mathscr{I}}(\mathbb{V}_1\cup\mathbb{V}_2)-\rho_U^{\mathscr{I}}(\mathbb{V}_1)\,\rho_U^{\mathscr{I}}(\mathbb{V}_2)$ is equal to

$$\sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_1\cup\mathbb{V}_2)}\ \sum_{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)} \Delta_{\mathbb{S}}^U(\mathbb{C})$$
$$-\left[\sum_{\mathbb{S}_1\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_1)}\ \sum_{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)} \Delta_{\mathbb{S}_1}^U(\mathbb{C})\right]$$
$$\times\left[\sum_{\mathbb{S}_2\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_2)}\ \sum_{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)} \Delta_{\mathbb{S}_2}^U(\mathbb{C})\right]$$

Due to the factorization property of $\Delta_{\mathbb{S}}^U$ formulated at the end of Proposition 5.3, some terms can be cancelled out. Since $\Delta_{\mathbb{S}}^U(\mathbb{C})=0$ when $\mathbb{C}\not\supset\mathbb{S}$ and $d(C_1\cup\operatorname{supp}\mathbb{S}_1,\ C_2\cup\operatorname{supp}\mathbb{S}_2)>1$ when $\mathbb{S}_1\cap\mathbb{V}_1\neq\varnothing$, $\mathbb{S}_2\cap\mathbb{V}_2\neq\varnothing$, $C_1\supset\operatorname{supp}\mathbb{S}_1$, $C_2\supset\operatorname{supp}\mathbb{S}_2$, $\|\mathbb{C}_q\|<\frac{1}{2}d(\operatorname{supp}\mathbb{V}_1,\ \operatorname{supp}\mathbb{V}_2)$ for $q=1, 2$, we have, denoting $d(\operatorname{supp}\mathbb{V}_1,\ \operatorname{supp}\mathbb{V}_2)$ by d, the inequalities

$$|\rho_U^{\mathscr{I}}(\mathbb{V}_1\cup\mathbb{V}_2)-\rho_U^{\mathscr{I}}(\mathbb{V}_1)\,\rho_U^{\mathscr{I}}(\mathbb{V}_2)|$$
$$\leqslant \sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_1\cup\mathbb{V}_2)}\ \sum_{\substack{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)\\ \|\mathbb{C}\|\geqslant d/2}} |\Delta_{\mathbb{S}}^U(\mathbb{C})|$$
$$+\left(\sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_1)}\ \sum_{\substack{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)\\ \|\mathbb{C}\|\geqslant d/2}} |\Delta_{\mathbb{S}}^U(\mathbb{C})|\right)$$
$$\times\left(\sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_2)}\ \sum_{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)} |\Delta_{\mathbb{S}}^U(\mathbb{C})|\right)$$
$$+\left(\sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_1)}\ \sum_{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)} |\Delta_{\mathbb{S}}^U(\mathbb{C})|\right)$$
$$\times\left(\sum_{\mathbb{S}\in\mathscr{A}^{\mathrm{co}}(U,\mathbb{V}_2)}\ \sum_{\substack{\mathbb{C}\in\mathscr{A}^{\mathrm{cl}}(U)\\ \|\mathbb{C}\|\geqslant d/2}} |\Delta_{\mathbb{S}}^U(\mathbb{C})|\right)$$
$$\leqslant 3\exp[-(\omega-t\log 2)\|\mathbb{V}_1\cup\mathbb{V}_2\|-\tfrac{1}{2}\omega d(\operatorname{supp}\mathbb{V}_1, \operatorname{supp}\mathbb{V}_2)]$$

The assertions (a)–(c) from Theorem 2(ii) follow from Proposition 3.4 for the cylinder V with any base.

6.1.2. We now prove the equality (2.1) for a cylinder U with a finite base. The Gibbs state $\mu_U^{\beta H}$ is denoted by μ in the following. Since $\mathscr{I}(U)$ is countable, we can write

$$\mu(A) = \int \mu(A \mid \mathbb{I})\, dP_U^{\mathscr{I}}(\mathbb{I}) \tag{6.2}$$

where

$$\mu(A \mid \mathbb{I}) = \frac{\mu(A \cap \{x \mid \mathbb{I}(x) = \mathbb{I}\})}{\mu(\{x \mid \mathbb{I}(x) = \mathbb{I}\})}$$

for μ-a.e. $\mathbb{I} \in \mathscr{I}(U)$. [In fact, $\mu(\{x \mid \mathbb{I}(x) = \mathbb{I}\}) > 0$ for any $\mathbb{I} \in \mathscr{I}(U)$.] We choose a fixed $\mathbb{I} = (I, x_I) \in \mathscr{I}(U)$ [such that $P_U^{\mathscr{I}}(\mathbb{I}) > 0$]. Let $x \in X$ be such that $B(x) = I$. Then $\mu(\cdot \mid \mathbb{I})$ is the Gibbs state in $U_1^0(I) \cup U_2^0(I) \cup (\bigcup \mathrm{Int}_\alpha I)^0$ with the boundary condition x (recall that $U^0 = \{i \in U \mid d(i, U^c) \geq R + 2\}$) because for any finite $\Lambda \subset U_1^0(I) \cup U_2^0(I)$ and f bounded and measurable we get

$$\begin{aligned}
\mu(f \mid \mathbb{I}) &= \mu(f \times \chi_{\mathbb{I}})/\mu(\chi_{\mathbb{I}}) \\
&= \left[\iint (f \times \chi_{\mathbb{I}})(z_\Lambda \times u_{\Lambda^c})\, \mu_\Lambda^{\beta H}(dz \mid u)\, \mu(du) \right] \Big/ \mu(\chi_{\mathbb{I}}) \\
&= \left\{ \int \left[\int f(z_\Lambda \times u_{\Lambda^c})\, \mu_\Lambda^{\beta H}(dz \mid u) \right] \chi_{\mathbb{I}}(u)\, \mu(du) \right\} \Big/ \mu(\chi_{\mathbb{I}}) \\
&= \iint f(z_\Lambda \times u_{\Lambda^c})\, \mu_\Lambda^{\beta H}(dz \mid u)\, \mu(du \mid \mathbb{I})
\end{aligned}$$

where $\chi_{\mathbb{I}} = \chi_{\{\tilde{x} \mid \mathbb{I}(\tilde{x}) = \mathbb{I}\}}$.

Finally, we have

$$\begin{aligned}
&\mu(\{z \in X \mid z_{(U_1^0(I) \cup U_2^0(I))^c} = x_{(U_1^0(I) \cup U_2^0(I))^c}\} \mid \mathbb{I}) \\
&\quad = \mu(\chi_{\mathbb{I}} \mid \mathbb{I}) = 1
\end{aligned}$$

We use a direct generalization of Lema 2.1:

Lemma 2.1′. Let $V \subset \mathbb{Z}^\nu$ be nonempty, $z \in X$, and $V = \bigcup V_\alpha$, where V_α are such that $d(V_\alpha, V_{\alpha'}) > R$ for $\alpha \neq \alpha'$. Let further (i) V_α be finite, or (ii) V be a subset of a cylinder set with a finite base.

Then there is the only Gibbs state $\mu_V^{\beta H}(\cdot \mid z_{V^c})$ in V with the boundary condition z_{V^c} and we have

$$\mu_V^{\beta H}(\cdot \mid z_{V^c}) = \bigotimes_\alpha \pi_{V_\alpha} \mu_{V_\alpha}^{\beta H}(\cdot \mid z_{V_\alpha^c}) \otimes \pi_{(\bigcup V_\alpha)^c}\, \delta_z(\cdot)$$

Here δ_z is the Dirac probability with $\delta_z(\{z\}) = 1$ and by π_V we indicate the projection of a measure onto X_V.

Let us consider now the uniquely determined Gibbs states $\mu_{U_q^0}^{\beta H}(\cdot \mid x^q)$ in $U_q^0 \equiv U_q^0(I)$ with the boundary condition $x^q_{(U_q^0)^c}$ for $q = 1, 2$, and $\mu_{(\bigcup \mathrm{Int}_\alpha I)^c}(\cdot \mid x_{\mathbb{I}})$ in $(\bigcup \mathrm{Int}_\alpha I)^0$ with the boundary condition $(x_{\mathbb{I}})_{((\bigcup \mathrm{Int}_\alpha I)^0)^c}$. Using (ii) from the preceding lemma the Gibbs state μ is uniquely determined, it is equal to

$$\pi_{U_1^0} \mu_{U_1^0}^{\beta H}(\cdot \mid x_{(U_1^0)^c}) \otimes \pi_{U_2^0} \mu_{U_2^0}^{\beta H}(\cdot \mid x_{(U_2^0)^c}) \otimes \pi_{(U_1^0 \cup U_2^0)^c} \mu_{(\bigcup \mathrm{Int}_\alpha I)^0}(\cdot \mid x)$$

Since obviously

$$\pi_{U_q^0} \mu_{U_q^0}^{\beta H}(\cdot \mid x_{(U_q^0)^c}) = \pi_{U_q^0} \mu_{U_q^0}^{\beta H}(\cdot \mid x^q_{(U_q^0)^c})$$

for $q = 1, 2$, one has

$$\mu(\cdot \mid \mathbb{I}) = \pi_{U_1^0} \mu_{U_1^0}^{\beta H}(\cdot \mid x^2) \otimes \pi_{U_2^0} \mu_{U_2^0}^{\beta H}(\cdot \mid x^2) \otimes \pi_{(U_1^0 \cup U_0^0)^c} \mu_{(\bigcup \mathrm{Int}_\alpha I)^0}^{\beta H}(\cdot \mid x_{\mathbb{I}}) \qquad (6.3)$$

From Proposition 3.4 we know that

$$\mu_{U_q^0}^{\beta H}(\cdot \mid x^q) = \int_{\mathscr{K}_q^{\mathrm{e}}(U_q)} \mu(\cdot \mid \theta_q) \; P_{q, U_q}^{\mathrm{e}}(d\theta_q), \qquad q = 1, 2$$

The restriction of the configuration $x_{I^0(\theta_q)^0}$ (defined uniquely by the requirement that the external contours of x be equal to θ_q) to $(U_q^0)^c$ is equal to $x^q_{(U_q^0)^c}$ whenever $\theta_q \in \mathscr{K}_q^{\mathrm{e}}(U_q)$, and therefore

$$\pi_{U_q^0} \mu_{U_q^0}^{\beta H}(\cdot \mid x^q) = \int_{\mathscr{K}_q^{\mathrm{e}}(U_q)} \pi_{U_q^0} \mu(\cdot \mid \theta_q) \, P_{q, U_q}^{\mathrm{e}}(d\theta_q), \qquad q = 1, 2 \qquad (6.4)$$

Since $\mathbb{I}$ is firmly chosen and $\theta_q \in \mathscr{K}_q^{\mathrm{e}}(U_q(I))$, $q = 1, 2$, we have, according to Lemma 2.1′(i),

$$\mu(\cdot \mid \theta_1, \theta_2, \mathbb{I})$$
$$= \pi_{U_1^0} \mu(\cdot \mid \theta_1) \otimes \pi_{U_2^0} \mu(\cdot \mid \theta_2) \otimes \pi_{(U_1^0 \cup U_2^0)^c} \mu_{(\bigcup \mathrm{Int}_\alpha \mathbb{I})^0}(\cdot \mid x_{\mathbb{I}}) \qquad (6.5)$$

Combining (6.2)–(6.5), one gets the equality (2.1) for a cylinder U with a finite base.

6.2. Let V be a cylinder (with a not necessarily finite base). The inequality (i)(b) already proved for cylinder sets with finite bases implies that the limit over cylinders with finite bases ordered by inclusion exists, namely $\rho_V^{\mathscr{I}} = \lim_{U \nearrow V} \rho_U^{\mathscr{I}}$. This limit is obviously nonnegative and satisfies (a)–(c) from Theorem 2(i).

This finishes the proof of part (i) of Theorem 2. Part (ii) is known from Proposition 3.4.

6.2.1. One realizes easily that, according to Lemma 2.2, the probabilities $P_U^{\mathscr{I}}$ can be understood as probabilities on $\mathscr{W}^{co}$, in fact on $\mathscr{W}^{a} \subset \mathscr{W}^{co}$ for cylinders U with finite bases. It follows from the existence of the limit $\rho_V^{\mathscr{I}} = \lim_{U \nearrow V} \rho_U^{\mathscr{I}}$ that $P_U^{\mathscr{I}}$ converge weakly to $P_V^{\mathscr{I}}$. Namely, the probability $P_V^{\mathscr{I}}$ is defined uniquely by its values on sets of the form

$$\mathscr{M}_{M, \mathbb{V}} = \{\mathbb{V}' \in \mathscr{W}(V) = 2^{\mathrm{W}(V)} \mid \mathbb{V}' \cap M = \mathbb{V}\}$$

for finite sets $M \subset \mathscr{W}(V)$ and $\mathbb{V} \subset \mathscr{W}(V)$. Since

$$\mathscr{M}_{M, \mathbb{V}} = \mathscr{M}_{\mathbb{V}} \Big\backslash \bigcup_{\mathrm{w} \in M \backslash \mathbb{V}} \mathscr{M}_{\mathbb{V} \cup \{\mathrm{w}\}}$$

where $\mathscr{M}_{\mathbb{V}} = \{\mathbb{V}' \in \mathscr{W}(V) \mid \mathbb{V}' \supset \mathbb{V}\}$ for $\mathbb{V} \subset M$, the probability $P_V^{\mathscr{I}}(\mathscr{M}_{M, \mathbb{V}})$ is defined by the expression

$$\sum_{\mathbb{V}' \subset M \backslash \mathbb{V}} (-1)^{|\mathbb{V}'|} P_V^{\mathscr{I}}(\mathscr{M}_{\mathbb{V} \cup \mathbb{V}'})$$

Note

$$\bigcap_{\mathrm{w} \in \mathbb{V}'} \mathscr{M}_{\mathbb{V} \cup \{\mathrm{w}\}} = \mathscr{M}_{\mathbb{V} \cap \mathbb{V}'} \qquad \text{for } \mathbb{V} \cup \mathbb{V}' \text{ compatible}$$

$$= \varnothing \qquad \text{otherwise}$$

Since we have already proved that $\rho_u^{\mathscr{I}} \to \rho_V^{\mathscr{I}}$, we know that $P_V^{\mathscr{I}} = \lim_{U \nearrow V} P_U^{\mathscr{I}}$ weakly in $\mathscr{W}(V)$. It is simple to notice that $P_V^{\mathscr{I}}(\mathscr{W}^{co}(V)) = 1$ [the set $\mathscr{W}(V) \backslash \mathscr{W}^{co}(V)$ is covered by the countable union of sets of families of walls that are "incompatible at $i, j \in \mathbb{Z}^\nu$ for i, j neighbors"].

According to the inequality (i)(a), we can deduce that $P_V^{\mathscr{I}}(\mathscr{W}^{a}(V)) = 1$ similarly as we deduced the corresponding fact for contours in Theorem B.2 (see Appendix B). We need only that the expression $2\omega - t \log 2$ from the exponent in the estimate (i)(a) of Theorem 2 is greater than $\log(c)$ for the maximal ω allowed in Proposition 5.1, i.e.,

$$2 \min(\beta\rho_1 - c_5, \beta\rho_2 - c_6) > t \log 2 + \log(2c)$$

This follows from the assumptions of Theorem 2.

Lemma 2.2 implies that the probability $P_V^{\mathscr{I}}$ is defined on the space $\mathscr{I}^{a}(V) \subset \mathscr{I}(V)$ of admissible interfaces.

6.2.2. It remains to prove equality (2.1). Since we know already that (2.1) holds for cylinders U with finite bases, it suffices to prove that

$$\int P_U^{\mathscr{I}}(d\mathbb{I}) \int \mu(\varphi \mid \mathbb{I}, \theta_1, \theta_2)\, P^{\mathrm{e}}_{1, U_1(I)}\, d\theta_1)\, P^{\mathrm{e}}_{2, U_2(I)}(d\theta_2)$$

converges to the right-hand side of (2.1) for any cylindrical function φ. Let us suppose that φ lives on a finite subset Λ of $\mathbb{Z}^\nu$. We consider the function

$$f_U^k(\mathbb{I}) = \chi(\mathbb{I}) \int_{\Theta_1 \times \Theta_2} \mu(\varphi \mid \mathbb{I}, \theta_1, \theta_2)\, P^{\mathrm{e}}_{1, U_1(I)}(d\theta_1)\, P^{\mathrm{e}}_{2, U_2(I)}(d\theta_2)$$

where

$$\Theta_q = \{\theta_q \in \mathscr{K}^{\mathrm{e}}_q(U_q(I)) | (\Gamma \cap \Lambda_R \neq \varnothing, \gamma \in \theta_q) \Rightarrow |\Gamma| \leqslant k\}$$

and $\chi(\mathbb{I}) = 1$ if $\mathbb{w} \in \mathbb{W}(\mathbb{I})$, $W \cap \Lambda_R \neq \varnothing \Rightarrow |W| \leqslant k$ and $\chi(\mathbb{I}) = 0$ otherwise.

We use the same definition for $f_U^\infty(\mathbb{I})$.

It can be deduced from the estimates (a) in (i) and (ii) that $\int P_U^{\mathscr{I}}(d\mathbb{I})\, f_U^k(\mathbb{I})$ converges to $\int P_U^{\mathscr{I}}(d\mathbb{I})\, f_U^\infty(\mathbb{I})$ uniformly in U.

Thus, it suffices to prove that $\int P_U^{\mathscr{I}}(d\mathbb{I})\, f_U^k(\mathbb{I})$ converges to $\int P_V^{\mathscr{I}}(d\mathbb{I})\, f_V^k(\mathbb{I})$ for "U with finite bases converging to V accordingly to the order by inclusion."

We can express the integral $\int_{\Theta_1 \times \Theta_2} \cdots$ as a finite linear combination of products of correlations of the form $\rho^{\mathrm{e}}_{1, U_1(I)}(\theta_1)\, \rho^{\mathrm{e}}_{2, U_2(I)}(\theta_2)$, where θ_1, θ_2 are families of contours γ for which $|\Gamma| \leqslant k$, and $\Gamma \cap \Lambda_R \neq \varnothing$. [Notice that $\mu(\varphi \mid \mathbb{I}, \theta_1, \theta_2)$ is constant on the set of θ_1, θ_2 such that $U\{\Gamma \cap \Lambda_R \mid \gamma \in \theta_q, q = 1, 2\}$ is given.]

It follows from (i)(b) that these correlations converge for U converging to V uniformly in θ_1, θ_2 because $d(\Lambda_{R+k}, U \div V)$ converges to infinity for U converging to V and thus $f_U^k(\mathbb{I})$ converges to $f_V^k(\mathbb{I})$ uniformly in U.

We thus have

$$\int P_{\bar{U}}^{\mathscr{I}}(d\mathbb{I})\, f_U^k(\mathbb{I}) \quad \text{converges to} \int P_{\bar{U}}^{\mathscr{I}}(d\mathbb{I})\, f_V^k(\mathbb{I}) \tag{6.6}$$

uniformly in cylinders $\bar{U}$ with finite bases.

Similarly, for $\mathbb{I}'$ converging to $\mathbb{I}$ we have $f_V^k(\mathbb{I}')$ converges to $f_V^k(\mathbb{I})$ because $d(\Lambda_{R+k}, U_q(I') \div U_q(I))$ converges to infinity and we can use (i)(b).

Hence f_V^k is continuous (notice that it is not cylindrical because the probabilities $P^{\mathrm{e}}_{q, U_q(I)}$ depend on the change of I far from Λ) and

$$\int P_U^{\mathscr{I}}(d\mathbb{I})\, f_V^k(\mathbb{I}) \quad \text{converges to} \quad \int P_V^{\mathscr{I}}(d\mathbb{I})\, f_V^k(\mathbb{I}) \tag{6.7}$$

The properties (6.6), (6.7) imply that $\int P_U^{\mathscr{I}}(d\mathbb{I})\, f_U^k(\mathbb{I})$ converges to $\int P_V^{\mathscr{I}}(d\mathbb{I})\, f_V^k(\mathbb{I})$.

This finishes the proof of Theorem 2.

6.3. Proof of Theorem 1

We shall prove the extremality (i). The other statements (ii)–(iv) are an immediate consequence of Theorem 2.

Repeating the proof of Corollary 3.1 from Ref. 4b, we see that the extremality of the Gibbs state μ from Theorem 1 follows from the following result.

Proposition 6.2. Under the assumptions of Theorem 1, there exist constants $\kappa > 0$ and $A > 0$ such that

$$|\mu(\varphi_1 \varphi_2) - \mu(\varphi_1)\,\mu(\varphi_2)| \leqslant A\,|\Lambda_1|\,|\Lambda_2| \exp[-\kappa\beta d(\Lambda_1, \Lambda_2)]$$

whenever φ_1, φ_2 are Λ_1, Λ_2 cylinder functions such that $\|\varphi_1\| \leqslant 1$, $\|\varphi_2\| \leqslant 1$, respectively.

This is (a slightly reformulated version of) Proposition 3.2 from Ref. 4b. While we shall in principle follow its proof, we first have to fill a gap in it. Namely, the proof in Ref. 4b does not apply, e.g., if $\pi(\Lambda_1) \cap \pi(\Lambda_2) \neq \emptyset$ and this is exactly the case used in the proof of Corollary 3.1 stating extremality.

The gap may be filled with help of an estimate on the height of the interface proved in Ref. 12 and also in Ref. 4:

Lemma 6.3. Under the assumptions of Theorem 1, there exist constants $\bar{\alpha} > 0$ and $\bar{K} > \infty$ such that for all $i \in I_0$ one has

$$\mu(\{x \mid h_i(\mathbb{I}(x)) \geqslant N\}) \leqslant \bar{K} \exp(-\bar{\alpha}\beta N)$$

where $h_i(\mathbb{I}) = \sup\{d(j, I_0)) \mid j \in I,\ \pi(j) \ni i\}$.

Proof. Denoting by $\mathbb{W}_i(\mathbb{I})$ the set of all standard walls in $\mathbb{I}$ encircling the point i,

$$\mathbb{W}_i(\mathbb{I}) = \{\mathbb{w} \in \mathbb{W}(\mathbb{I}) \mid i \in \mathrm{Int}_{I_0}(\pi(W)) \cup \pi(W)\}$$

we clearly have

$$h_i(\mathbb{I}) \leqslant \sum_{\mathbb{w} \in \mathbb{W}_i(\mathbb{I})} |W| = \|\mathbb{W}_i(\mathbb{I})\|$$

Denoting further by $\mathscr{W}_i^{\mathrm{a}}$ the set of all admissible families $\mathbb{V}$ of standard walls such that every wall $\mathbb{w} \in \mathbb{V}$ encircles i ($i \in \mathrm{Int}_{I_0}(\pi(W)) \cup \pi(W)$), we have for the expectation with respect to the measure μ and any $\bar{\alpha} > 0$:

$$
\begin{aligned}
&E(\exp[\bar{\alpha}\beta h_i(\mathbb{I})]) \\
&\quad \leqslant E(\exp[\bar{\alpha}\beta \, \| W_i(\mathbb{I}) \|]) \\
&\quad \leqslant \sum_{\mathbb{V} \in \mathscr{W}_i^{\mathrm{a}}} [\exp(\bar{\alpha}\beta \, \| \mathbb{V} \|)] \, \rho^{\mathscr{I}}(\mathbb{V}) \\
&\quad \leqslant \sum_{\mathbb{V} \in \mathscr{W}_i^{\mathrm{a}}} \exp[-(\bar{\omega} - t \log 2 - \bar{\alpha}\beta) \, \| \mathbb{V} \|] \\
&\quad \leqslant \prod_{n=1}^{\infty} \left\{ 1 + \sum_{\substack{\mathbb{W} : \{\mathrm{w}\} \in \mathscr{W}_i^{\mathrm{a}} \\ |W| \geqslant n}} \exp[-(\bar{\omega} - t \log 2 - \bar{\alpha}\beta) \, |W|] \right\}
\end{aligned}
$$

The last expression is finite ($\leqslant \bar{K}$) for $\bar{\alpha}$ small enough. Hence, according to Chebyshev's inequality,

$$
\begin{aligned}
&\mu(\{\mathbb{I} \mid h_i(\mathbb{I}) \geqslant N\}) \\
&\quad = \mu(\{\mathbb{I} \mid \exp[\bar{\alpha}\beta h_i(\mathbb{I})] \geqslant \exp(\bar{\alpha}\beta N)\}) \\
&\quad \leqslant \bar{K} \exp(-\bar{\alpha}\beta N)
\end{aligned}
$$

Proof of Proposition 6.2. Consider δ, $N > 0$ and denote $D_a = (\Lambda_a)_\delta \cap (I_0)_N$, $E_a = \pi(D_a)$, $a = 1, 2$, where $(\Lambda_a)_\delta$ is the δ-neighborhood of Λ_a and $(I_0)_N$ is the N-neighborhood of I_0. Note that $d(E_1, E_2) \geqslant d(\Lambda_1, \Lambda_2) - 2N - 2\delta$. With the help of (2.1) one gets

$$
\begin{aligned}
&|\mu(\varphi_1 \varphi_2) - \mu(\varphi_1) \, \mu(\varphi_2)| \\
&\quad = \left| \int P^{\mathscr{I}}(d\mathbb{I}) \, \mu(\varphi_1 \varphi_2 \mid \mathbb{I}) - \int P^{\mathscr{I}}(d\mathbb{I}) \, \mu(\varphi_1 \mid \mathbb{I}) \int P^{\mathscr{I}}(d\mathbb{I}) \, \mu(\varphi_2 \mid \mathbb{I}) \right| \\
&\quad \leqslant \int P^{\mathscr{I}}(d\mathbb{I}) \, |\mu(\varphi_1 \varphi_2 \mid \mathbb{I}) - \mu(\varphi_1 \mid \mathbb{I}) \, \mu(\varphi_2 \mid \mathbb{I})| \\
&\qquad + \left| \int P^{\mathscr{I}}(d\mathbb{I}) \, \mu(\varphi_1 \mid \mathbb{I}) \, \mu(\varphi_2 \mid \mathbb{I}) - \int P^{\mathscr{I}}(d\mathbb{I}) \, \mu(\varphi_1 \mid \mathbb{I}) \int P^{\mathscr{I}}(d\mathbb{I}') \, \mu(\varphi_2 \mid \mathbb{I}') \right|
\end{aligned} \tag{6.8}
$$

We evaluate the first term by taking into account (6.3) and Proposition (3.4)(iii) by

$$
K |\Lambda_1 \cup \Lambda_2| \, \|\varphi_1\| \, \|\varphi_2\| \exp[-\alpha d(\Lambda_1, \Lambda_2)]
$$

In the second term we restrict the integrations to $\mathbb{I}$ corresponding to

$$
\mathbb{V} \in \mathscr{E} = \{\mathbb{V} \in \mathscr{W}^{\mathrm{a}} \mid \sup\{|h_i(\mathbb{I}(\mathbb{V}))| \; i \in \pi((\Lambda_1)_\delta \cup (\Lambda_2)_\delta\} \leqslant N\} \cap \mathscr{F}
$$

where

$$\mathscr{F} = \{\mathbb{V} \in \mathscr{W}^{\mathrm{a}} \mid \mathbb{w} \in \mathbb{V},\ [\pi(W) \cup \operatorname{Int}_{I_0} \pi(W)] \cap [E_1 \cup E_2] \neq \varnothing \\ \text{implies } |W| \leqslant k\}$$

Denoting

$$\mathbb{V}_a = \{\mathbb{w} \in \mathbb{V} \mid (\pi(W) \cup \operatorname{Int}_{I_0} \pi(W)) \cap E_a \neq \varnothing\}$$

and

$$\mathscr{E}_a = \{\mathbb{V} \in \mathscr{E} \mid \mathbb{V}_a = \mathbb{V}\}, \qquad \mathscr{F}_a = \{\mathbb{V} \in \mathscr{F} \mid \mathbb{V}_a = \mathbb{V}\}, \qquad a = 1, 2$$

note that for every $\mathbb{V} \in \mathscr{E}$ we have

$$\text{if} \quad \mathbb{w}_a \in \mathbb{V}_a,\ a = 1, 2, \quad \text{then} \quad d(\pi(W_1), \pi(W_2)) \geqslant d(E_1, E_2) - 2k \tag{6.9}$$

$$\text{if} \quad \mathbb{w} \in \mathbb{V} \backslash (\mathbb{V}_1 \cup \mathbb{V}_2), \quad \text{then} \quad d(I(\mathbb{V}) \cap \pi^{-1}(W), \Lambda_1 \cup \Lambda_2) \geqslant \delta \tag{6.10}$$

Note, moreover, that

$$P^{\mathscr{I}}(\mathscr{W}^{\mathrm{a}} \backslash \mathscr{E}) \leqslant |\pi(\Lambda_1 \cup \Lambda_2)|\, \delta^{\nu} \cdot \bar{K} \exp(-\bar{\alpha}\beta N) \tag{6.11}$$

and that for every $\mathbb{V} \in \mathscr{E}$, according to Proposition 3.4(iii) and (6.9), we have

$$|\mu(\varphi_a \mid \mathbb{I}(\mathbb{V})) - \mu(\varphi_a \mid \mathbb{I}(\mathbb{V}_a))| \leqslant K |\Lambda_a|\ \|\varphi_a\|\ e^{-\alpha\delta}, \qquad a = 1, 2$$

Hence, we have for the second term in (6.8) an estimate

$$3\,|\pi(\Lambda_1 \cup \Lambda_2)|\ \delta^{\nu} \bar{K} \exp(-\bar{\alpha}\beta N) + 2K(|\Lambda_1|\ \|\varphi_1\| + |\Lambda_2|\ \|\varphi_2\|) \exp(-\alpha\delta) \\ + \left| \int_{\mathscr{E}} P^{\mathscr{I}}(d\mathbb{I})\ \mu(\varphi_1 \mid \mathbb{I}(\mathbb{V}_1(\mathbb{I})))\, \mu(\varphi_2 \mid \mathbb{I}(\mathbb{V}_2(\mathbb{I}))) \right. \\ \left. - \int_{\mathscr{E}} P^{\mathscr{I}}(d\mathbb{I})\, \mu(\varphi_1 \mid \mathbb{I}(\mathbb{V}_1(\mathbb{I}))) \int_{\mathscr{E}} P^{\mathscr{I}}(d\mathbb{I}')\, \mu(\varphi_2 \mid \mathbb{I}(\mathbb{V}_2(\mathbb{I}'))) \right|$$

Again using (6.11), we estimate the last term by

$$3\,|\pi(\Lambda_1 \cup \Lambda_2)|\ \delta^{\nu} \bar{K} \exp(-\bar{\alpha}\beta N) \\ + \sum_{\substack{\mathbb{V}_1 \in \mathscr{F}_1 \\ \mathbb{V}_2 \in \mathscr{F}_2}} \mu(\varphi_1 \mid \mathbb{I}(\mathbb{V}_1))\, \mu(\varphi_2 \mid \mathbb{I}(\mathbb{V}_2))\, |P(\{\mathbb{U} \in \mathscr{F} \mid \mathbb{U}_1 = \mathbb{V}_1, \mathbb{U}_2 = \mathbb{V}_2\}) \\ - P^{\mathscr{I}}(\{\mathbb{U} \in \mathscr{F} \mid \mathbb{U}_1 = \mathbb{V}_1\})\, P^{\mathscr{I}}(\{\mathbb{U} \in \mathscr{F} \mid \mathbb{U}_2 = \mathbb{V}_2\})|$$

Realizing that the inequality (i)(c) from Theorem 2 can be generalized in a similar way as when the estimate (ii)(c) from Proposition 3.4 was extended to (ii)(c′) in its proof, we get for the last sum the estimate

$$3\,\|\varphi_1\|\,\|\varphi_2\|\,\exp\{-\tfrac{1}{2}\bar{\omega}[d(E_1,E_2)-2k]\}$$

7. PROOF OF THEOREM 3

Theorem 3 follows from:

Proposition 7.1. Under the assumptions of Theorem 3, we have for any cylinder $V=V_B$ with finite base

$$\lim_{U\nearrow V_B}\text{fin cyl}\log\frac{Z(U\mid y^{1,2};\beta H)}{Z(U_1(I_0)\backslash(I_0)_R\mid x^1)\,Z(U_2(I_0)\backslash(I_0)_R\mid x^2)}$$
$$=-\beta\sum_{\substack{A\subset(I_0)_R\\ A\cap V\neq\varnothing}}\varphi_A(y)+\sum_{\mathbb{C}\in\mathscr{A}^{\text{cl}}(\mathbb{A}(V))}\Psi^{V,T}(\mathbb{C})$$
$$+\sum_{q=1,2}\;\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\text{cl}}\\ \operatorname{supp}\mathbb{C}\cap(I_0\cap V_R)\neq\varnothing}}\Phi_q^T(\mathbb{C})\frac{|\operatorname{supp}\mathbb{C}\cap(V_R)_q|}{|\operatorname{supp}\mathbb{C}|}+\tilde{\Delta}$$

where $|\tilde{\Delta}|\leqslant[e^{-\omega}/(1-e^{-\omega})]\,|\partial B|$ with ω from Theorem 2, and Δ from Theorem 3 equal to

$$\Delta=\sum_{\substack{\mathbb{C}\in\mathscr{A}^{\text{cl}}\\ \pi(\mathbb{C})\ni i}}\Psi^{V,T}(\mathbb{C})\frac{|\pi(i)|}{|\pi(\operatorname{supp}\mathbb{C})|}$$
$$+\sum_{q=1}^{2}\;\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\text{cl}}\\ \operatorname{supp}\mathbb{C}\cap\pi(i)\neq\varnothing}}\Phi_q^T(\mathbb{C})\frac{|\pi(i)|}{|\pi(\operatorname{supp}\mathbb{C})|}$$

The $\psi^{V,T}$ were introduced in Proposition 5.3 and Φ_q^T in Section 3.

Proof. According to Lemmas 4.2, 4.3, and 4.9, we get

$$\lim_{U\nearrow V}\text{fin cyl}\log\frac{Z(U\mid y^{1,2};\beta H)}{Z(U_1(I_0)\backslash(I_0)_R\mid x^1)\,Z(U_2(I_0)\backslash(I_0)_R\mid x^2)}$$
$$=\log\tilde{\mathscr{Z}}(\mathbb{A}(V)\mid\Psi^V)$$
$$+\lim_{U\nearrow V_B}\text{fin cyl}\log\frac{N(U\mid y;\beta H)}{Z(U_1(I_0)\backslash(I_0)_R\mid x^1)\,Z(U_2(I_0)\backslash(I_0)_R\mid x^2)}$$

The second term is, according to Lemmas 4.3 and 3.1

$$\lim_{U\nearrow V}\operatorname{fin}\operatorname{cyl}\left\{p(\beta H)|U_R|+\beta\sum_{A\subset U^c}\varphi_A(y)\frac{|A\cap U_R|}{|A|}\right.$$
$$-\sum_{q=1}^{2}\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \mathbb{C}_*q}}\Phi_q^T(\mathbb{C})\frac{|C\cap U_R|}{|C|}-p(\beta H)|I_0\cap U_R|-\beta E_{I_0\cap U_R}(\mathbb{I}_0)$$
$$-\sum_{q=1}^{2}\left[\beta\sum_{A\subset[U_q(I_0)\backslash(I_0)_R]^c}\varphi_A(x^q)\frac{|A\cap(U_R)_q(I_0)|}{|A|}-\beta e(x^q)|(U_R)_q(I_0)|\right.$$
$$\left.\left.+\log\theta((U_R)_q(I_0)\,|\,x^q;\beta H)\right]\right\}$$

Taking into account that $\beta e(x^q)=p(\Phi_q)-p(\beta H)$ for $q=1,2$ and using Proposition 3.2(ii) combined with Theorem B.1, we get for this term

$$\lim_{U\nearrow V}\operatorname{fin}\operatorname{cyl}\left\{\beta\left[\sum_{A\subset U^c}\varphi_A(y)\frac{|A\cap U_R|}{|A|}\right.\right.$$
$$\left.-\sum_{q=1}^{2}\sum_{A\subset[U_q(I_0)\backslash(I_0)_R]^c}\varphi_A(x^q)\frac{|A\cap(U_R)_q(I_0)|}{|A|}-E_{I_0\cap U_R}(\mathbb{I}_0)\right]$$
$$+\sum_{q=1}^{2}\left[\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \operatorname{supp}\mathbb{C}\cap((U_R)_q)^c\neq\varnothing}}\Phi_q^T(\mathbb{C})\frac{|\operatorname{supp}\mathbb{C}\cap(U_R)_q|}{|\operatorname{supp}\mathbb{C}|}\right.$$
$$\left.\left.-\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \mathbb{C}_*q}}\Phi_q^T(\mathbb{C})\frac{|\operatorname{supp}\mathbb{C}\cap U_R|}{|\operatorname{supp}\mathbb{C}|}\right]\right\}$$
$$=-\beta\sum_{\substack{A\subset(I_0)_R\\ A\cap V\neq\varnothing}}\varphi_A(y)+\sum_{q=1}^{2}\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \operatorname{supp}\mathbb{C}\cap(I_0\cap V_R)\neq\varnothing}}\Phi_q^T(\mathbb{C})\frac{|\operatorname{supp}\mathbb{C}\cap(V_R)_q|}{|\operatorname{supp}\mathbb{C}|}$$
$$-\sum_{q=1}^{2}\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \mathbb{C}_*q\\ \operatorname{supp}\mathbb{C}\cap(I_0\cap V_R)\neq\varnothing}}\Phi_q^T(\mathbb{C})\frac{|\operatorname{supp}\mathbb{C}\cap V_R|}{|\operatorname{supp}\mathbb{C}|}$$

Using Proposition 5.3 for the expansion of $\log\tilde{\mathscr{Z}}(\mathbb{A}(V)\,|\,\psi^V)$ and taking the third term above for $\tilde{\Delta}$, we finally get the statement of the proposition. The indicated form of Δ then follows in a straightforward manner.

APPENDIX A. THE GEOMETRICAL STRUCTURE OF INTERFACES

Our aim in this Appendix is to prove Lemmas 4.1, 4.8, and 2.2. Most methods and formulations are modifications of geometrical lemmas from Ref. 3 or Ref. 16.

We suppose $v \geqslant 3$, though in some lemmas it is enough to suppose $v \geqslant 2$, as will be indicated.

We begin by recalling a useful lemma,[16] which, among other things, implies that contours are q-contours for some $q \in \{1,..., r\}$ as mentioned in Section 2.2. It will serve several times in the analysis of interfaces.

Lemma A.1. Let $v \geqslant 2$ and $M \subset \mathbb{Z}^v$ and $\mathbb{Z}^v \backslash M$ be R-connected ($R = 1, 2,...$). Then $\partial M = \{i \in M \mid d(i, \mathbb{Z}^v \backslash M) = 1\}$ is R-connected.

For the reader's convenience we reproduce here the proof from Ref. 16:

Put

$$A = \{x \in \mathbb{R}^v \mid d(x, M) < R/2 + 1/3\}$$

$$B = \{x \in \mathbb{R}^v \mid d(x, \mathbb{Z}^v \backslash M) < R/2 + 1/3\}$$

According to Ref. 17, Chapter 8, §52, II, Theorem 2 and Ref. 17, Chapter 9, §59, II, Theorem 11, the intersection $A \cap B$ is connected. We choose $x_1,..., x_p$ in $A \cap B$ such that $\rho(x_j, x_{j+1}) < 1/3$, $\rho(a, x_1) < R/2 + 1/3$, and $\rho(b, x_p) < R/2 + 1/3$ whenever $a, b \in \partial M$. Let us find $y_j \in \partial M$, $j = 2,..., p-1$, such that $\rho(y_j, x_j) < R/2 + 1/3$. The sequence a, $y_2,..., y_{p-1}$, b proves that a, b belong to the same (and thus the only) R-component of ∂M.

It is often useful to realize that the components of the complement of an R-connected set always satisfy the assumptions of Lemma A.1.

Proof of Lemma 4.1. It is obvious that there are at least one and at most two infinite components of $B(x)$. Let us suppose that the complement of one of those infinite components (say I) of $B(x)$ has a unique infinite R-component (say I^c_∞). According to Lemma A.1, we know that ∂I^c_∞ is R-connected. Since $\partial I^c_\infty \cap B(x) = \varnothing$, we have

$$x_{\partial I^c_\infty} = x^q_{\partial I^c_\infty}$$

for some $q \in \{1,..., r\}$, but this is obviously impossible.

To prove the other lemmas, we need some observations about ceilings and walls. Sometimes we shall deal with I_0 [and its subsets of the form $S = \pi^{-1}(S) \cap I_0$] as with $\mathbb{Z}^{v-1}$ (and its subsets) in an obvious sense.

Lemma A.2. Let $\nu \geqslant 2$ and let I be the support of an interface.

(a) Let D be a ceiling column. Then one has $h(D) = h(\pi^{-1}(i) \cap I)$ for any $i \in I$ such that $d(i, D) \leqslant R$. In particular, the ceiling columns that are contained in the same ceiling C have the same *height* $h(C)$.

(b) Let D be a conected subset of I_0 such that $\pi^{-1}(D) \cap I$ is contained in the union of all ceilings. Then $\pi^{-1}(D) \cap I$ is contained in a sole ceiling.

(c) Let C be a ceiling of I, and G be a component of $I_0 \backslash \pi(C)$ in I_0. Then $\pi^{-1}(G) \cap I$ is connected.

Proof:

(a) Let D be a ceiling column. The assertion follows from the fact that the heighest and the lowest hypercubes B such that $B \cap \pi^{-1}(D) = B \cap D \neq \varnothing$ are bad ones.

(b) Due to (a), all ceiling columns from $\pi^{-1}(D) \cap I$ are of the same height and thus belong to the same ceiling.

(c) The case $\nu = 2$ is simple and we omit the proof. Let $\nu > 2$. Let $i, j \in \pi^{-1}(G) \cap I$ and $k_1, ..., k_p \in I$ be such that $k_1 = i$, $k_p = j$, $\rho(k_l, k_{l+1}) \leqslant 1$. Let l_0 be the smallest index with $k_{l_0} \in C$ and l_1 be the largest index such that $k_{l_1} \in C$. Then $k_{l_0 - 1}$ and $k_{l_1 + 1}$ are elements of $\pi^{-1}(\partial(\pi(G))) \cap I$. Lemma A.1 implies that $\partial\pi(G) \cap I_0$ is connected in I_0 and the columns $\pi^{-1}(k) \cap I$ are of the same height by (a) for all $k \in \partial\pi(G) \cap I_0$.

Lemma A.3. (a) Let $\mathbb{w} = (W, x_W)$ be a wall of an interface $\mathbb{I} = (I, x_I)$. Then

$$\pi^{-1}\{i \in \mathbb{Z}^\nu \mid d(i, W) \leqslant 1\} \cap \bigcup_{\tilde{\mathbb{w}} \in \mathbb{W}(\mathbb{I})} \tilde{W} = W$$

(b) Let G be any component of $I_0 \backslash \pi(W)$ in I_0. Then $\pi^{-1}(\partial G \cap I_0) \cap I$ is contained in a sole ceiling C and $\pi^{-1}(i) \cap I$ is a column of height $h(C)$ whenever $i \in \partial G^c \cap I_0$.

Proof:

(a) The set $G_0 = \pi(\{i \in \mathbb{Z}^\nu \mid d(i, W) \leqslant 1\})$ is finite and connected in I_0. According to Definition 2.2 and Lemma A.2(c), the set $\pi^{-1}(G) \cap I$ is also finite and connected. Let us consider any ceiling C for which $\pi(C) \cap G_0 \neq \varnothing$. Lemma A.2(c) implies that $\pi^{-1}(G_1) \cap I$ is connected, where G_1 is the component of $I_0 \backslash \pi(C)$ containing $\pi(W)$. Removing all ceilings that intersect G_0, we get a connected subset of $\pi^{-1}(G_0) \cap I$ which contains W and does not intersect any ceiling of I. This implies (Definition 2.3) that the resulting set is identical to W.

(b) The set $\pi(W)$ is conected; thus, Lemma A.1 applies to $I_0\backslash G$ in I_0 and $\partial G\cap I_0$ is connected. Lemma A.3(a) implies that $\pi^{-1}(\partial G\cap I_0)\cap I$ is contained in the union of all ceilings and Lemma A.2(b) implies that it is contained in an only ceiling. The other assertion follows from Lemma A.2(a).

Recall that $\nu>2$ if not specified otherwise!

Proof of Lemma 2.2:

(a) We use Lemma A.3(b) for the only infinite component G of the complement of the support W of $\mathbb{w}$. We may denote the height of the ceiling from A.3(b) by $h(W)$ and show that $T_{-h(W)}\mathbb{w}$ is standard, applying Lemma A.3(b) to the other components of $\pi(W)^c$ in I_0.

(b) Lemma A.3(a) implies that $\mathbb{W}(\mathbb{I})\in\mathscr{W}^{\text{co}}$. Let $\mathbb{V}\in\mathscr{W}^{\text{a}}$. We know that there are finite external walls in $\mathbb{V}$ by the definition of $\mathscr{W}^{\text{a}}$. Using Lemma A.3, we construct easily an interface having the external wall $\mathbb{E}(\mathbb{V})$ of $\mathbb{V}$ as its only walls. The construction of $\mathbb{I}(\mathbb{V})$ can be completed by induction, continuing with external walls of $\mathbb{V}\backslash\mathbb{E}(\mathbb{V})$, etc. It is easy to realize that the construction gives the only possible interface with prescribed walls $\mathbb{V}$.

Part (a) of Lemma 4.8 is proved by the following:

Lemma A.4. Let $\mathbb{a}$ be an aggregate of an admissible triplet $\mathbb{T}$. Then

$$\pi^{-1}\left(\partial\left(\bigcap_{\mathbb{w}\in\mathbb{E}(\mathbb{a}_0)}\text{Ext}_{I_0}W\right)\right)\cap I(\mathbb{T})$$

is contained in one ceiling of $\mathbb{I}(\mathbb{T})$.

Proof. Let G be the only infinite component of

$$I_0\backslash\pi\left(\left(\bigcup_{\mathbb{w}\in\mathbb{a}_0}W\right)\cup\left(\bigcup_{\mathbb{C}\in\mathbb{a}_1\cup\mathbb{a}_2}C\right)\right)$$

Then $(\partial G)\cap I_0$ is connected in I_0 and $\pi^{-1}(\partial G\cap I_0)\cap I(\mathbb{T})$ is contained in one ceiling, by Lemma A.3(b). The set

$$F=[(I_0\backslash G)\cup\partial G]\cap\left(\bigcap_{\mathbb{w}\in\mathbb{a}_0}\text{Ext}_{I_0}W\right)$$

is connected and $F\cap\pi(W)=\emptyset$ for every $\mathbb{w}\in\mathbb{T}_0$. Here $\pi^{-1}(F)\cap I(\mathbb{T})$ is contained in one ceiling the statement folows, since

$$F\supset\partial\left(\bigcap_{\mathbb{w}\in\mathbb{E}(\mathbb{a}_0)}\text{Ext}_{I_0}W\right)$$

Proof of Lemma 4.8(b) may be carried out in a similar way to that of Lemma 2.2(b), considering whole aggregates instead of separate walls.

Remark. We could prove elementarily that the number of standard aggregates $\mathbb{a}$ with $i \in A$ and $\|\mathbb{a}\| = n$ is less than $\bar{c}^n$ for some constant $\bar{c}$ depending on v only. This estimate would yield an alternate proof of Lemma 5.2.

APPENDIX B. CONTOUR MODELS

We summarize here some rather standard[18,19] statements about contour (polymer) models. Since we shall use them in a situation where the role of contours is played by more complicated objects (namely, aggregates of walls and clusters; cf. Section 4.1), it is useful to introduce contour models in a more abstract form. In stating Theorem B.1, we closely follow Ref. 20.

Let us consider a countable set $\mathbb{K}$, the elements of which will be called contours. Let $\iota \subset \mathbb{K} \times \mathbb{K}$ be a reflexive and symmetric relation; pairs $(\gamma_1, \gamma_2) \in \iota$ (denoted also $\gamma_1 \,\iota\, \gamma_2$) will be called incompatible, while they will be called compatible if $(\gamma_1 . \gamma_2) \notin \iota$. By $\mathscr{K}^{\mathrm{co}}(\mathscr{K}^{f,\mathrm{co}})$ we denote the family of (finite) sets $\partial \subset \mathbb{K}$ consisting of mutually compatible contours. Considering a contour functional Φ: $\mathbb{K} \to \mathbb{C}$, we denote $\Phi(\partial) = \prod_{\gamma \in \partial} \Phi(\gamma)$ for each $\partial \in \mathscr{K}^f$, $\Phi(\varnothing) = 1$. If $\mathbb{L} \subset \mathbb{K}$ is a finite subset, the partition function $\mathscr{Z}(\mathbb{L}; \Phi)$ is defined by

$$\mathscr{Z}(\mathbb{L}; \Phi) = \sum_{\partial \in \mathscr{K}^{\mathrm{co}}(\mathbb{L})} \Phi(\partial)$$

where $\mathscr{K}^{\mathrm{co}}(\mathbb{L}) = \{\partial \in \mathscr{K}^{\mathrm{co}}, \partial \subset \mathbb{L}\}$. The correlation function $\rho_{\mathbb{L}}(\partial; \Phi)$ of ∂ in $\mathbb{L}$ is defined by

$$\rho_{\mathbb{L}}(\partial; \Phi) = \left[\sum_{\partial' : \partial \subset \partial' \in \mathscr{K}^{\mathrm{co}}(\mathbb{L})} \Phi(\partial')\right] \Big/ \mathscr{Z}(\mathbb{L}; \Phi)$$

Note that $\rho_{\mathbb{L}}(\partial; \Phi) = 0$ whenever $\partial \not\subset \mathbb{L}$. We often omit Φ in $\mathscr{Z}(\mathbb{L}; \Phi)$ and $\rho_{\mathbb{L}}(\partial; \Phi)$. The tacit assumption $\mathscr{Z}(\mathbb{L}) \neq 0$ needed for the definition of $\rho_{\mathbb{L}}$ will be always true under the hypothesis of Theorem B.1. We denote by $\mathscr{K}^f(\mathbb{L})$ the family of all finite subsets of $\mathbb{L} \subset \mathbb{K}$, $\mathscr{K}^f = \mathscr{K}^f(\mathbb{K})$. If $\mathbb{C} \in \mathscr{K}^f$, we denote $\mathbb{C}^c = \mathbb{K} \setminus \mathbb{C}$, $|\mathbb{C}|$ the number of contours in $\mathbb{C}$, and write $\mathbb{C} \,\iota\, \gamma$ whenever $\gamma \in \mathbb{K}$ and there is $\gamma' \in \mathbb{C}$ such that $\gamma \,\iota\, \gamma'$. We call $\mathbb{C} \in \mathscr{K}^f$ a *cluster* if it is not decomposable into two nonempty sets, $\mathbb{C} = \mathbb{C}_1 \cap \mathbb{C}_2$, such that every pair $\gamma_1 \in \mathbb{C}_1$, $\gamma_2 \in \mathbb{C}_2$ is compatible (i.e., such that $\mathbb{C}_1 \times \mathbb{C}_2 \cap \iota = \varnothing$). The set of all clusters will be denoted $\mathscr{K}^{\mathrm{cl}}$.

If $\mathscr{M}$ is a contractible set of contour functionals and $\mathscr{Z}(\mathbb{L}; \Phi) \neq 0$ for every $\Phi \in \mathscr{M}$ and every $\mathbb{L} \in \mathscr{K}^f$, as is the case in the following theorem, then there is a unique continuous branch of logarithm for which $\log \mathscr{Z}(\mathbb{L}; \Phi = 0) = 0$ [N.B.: $\mathscr{Z}(\mathbb{L}; \Phi = 0) = 1$ for every $\mathbb{L} \in \mathscr{K}^f$]. We always take $\log \mathscr{Z}(\mathbb{L}; \Phi)$ in this sense

Theorem B.1. Let functions a: $\mathbb{K} \to [0, \infty)$, l: $\mathbb{K} \to [0, \infty)$, and Φ: $\mathbb{K} \to \mathbb{C}$ and a number $\omega \geq 0$ be such that

$$\sum_{\gamma' : \gamma' \imath \gamma} \exp[a(\gamma') + \omega l(\gamma')] \, |\Phi(\gamma')| \leq a(\gamma) \tag{B1}$$

for each $\gamma \in \mathbb{K}$. Then $\mathscr{Z}(\mathbb{L}; \Phi) \neq 0$ for each finite $\mathbb{L} \subset \mathbb{K}$ and:

(i) There exists a unique function Φ^T: $\mathscr{K}^f \to \mathbb{C}$ such that

$$\log \mathscr{Z}(\mathbb{L}) = \sum_{\mathbb{C} \in \mathscr{K}^f(\mathbb{L})} \Phi^T(\mathbb{C}) \tag{B2}$$

for each finite $\mathbb{L} \subset \mathbb{K}$. Moreover, the function Φ^T is given by the formula

$$\Phi^T(\mathbb{C}) = \sum_{\mathbb{B} \in \mathbb{C}} (-1)^{|\mathbb{C}| - |\mathbb{B}|} \log \mathscr{Z}(\mathbb{B}) \tag{B3}$$

$$\sum_{\mathbb{C} \imath \gamma} |\Phi^T(\mathbb{C})| \, e^{\omega l(\mathbb{C})} \leq a(\gamma) \tag{B4}$$

for each $\gamma \in \mathbb{K}$ and with $l(\mathbb{C}) \leq \sum_{\gamma \in \mathbb{C}} l(\gamma)$. We have $\Phi^T(\mathbb{C}) = 0$ whenever $\mathbb{C} \notin \mathscr{K}^{\text{cl}}$.

(ii) For every $\partial \in \mathscr{K}^{f, \text{co}}$ there exists a unique function Δ_∂: $\mathscr{K}^f \to \mathbb{C}$ such that

$$\rho_{\mathbb{L}}(\partial) = \sum_{\mathbb{C} \subset \mathbb{L}} \Delta_\partial(\mathbb{C}) \tag{B5}$$

for each finite $\mathbb{L} \subset \mathbb{K}$. Moreover,

$$\Delta_\partial(\mathbb{C}) = \sum_{\mathbb{B} \subset \mathbb{C}} (-1)^{|\mathbb{C}| - |\mathbb{B}|} \rho_{\mathbb{B}}(\partial) \tag{B6}$$

for each finite $\mathbb{C} \subset \mathbb{K}$ and

$$\sum |\Delta_\partial(\mathbb{C})| \, e^{\omega l(\mathbb{C})} \leq e^{a(\partial) + \omega l(\partial)} \, |\Phi(\partial)| \tag{B7}$$

with $a(\partial) = \sum_{\gamma \in \partial} a(\gamma)$. We have $\Delta_\varnothing(\varnothing) = 1$ and $\Delta_\varnothing(\mathbb{C}) = 0$ for $\mathbb{C} \neq \varnothing$. The function Δ_∂ satisfies a factorization property:

$$\Delta_\partial(\mathbb{C}) = \Delta_{\partial_1}(\mathbb{C}_1) \, \Delta_{\partial_2}(\mathbb{C}_2) \tag{B8}$$

whenever $\partial=\partial_1\cup\partial_2$, $\mathbb{C}=\mathbb{C}_1\cup\mathbb{C}_2$, and all contours from $\mathbb{C}_1\cup\partial_1$ are compatible with those from $\mathbb{C}_2\cup\partial_2$.

Remark. When referring to above theorem we shall also use the following straightforward generalizations of the assertions (i) and (ii):

(i′) If $\mathbb{B}\subset\mathbb{K}$ is finite and $\mathbb{L}(\mathbb{B})=\{\gamma\in\mathbb{K}\mid\gamma\imath\mathbb{B}\}$, then $\mathscr{Z}(\mathbb{L}(\mathbb{B}))$ converges and

$$\log\mathscr{Z}(\mathbb{L}(\mathbb{B}))=\sum_{\mathbb{C}\in\mathscr{K}^{\mathrm{cl}}(\mathbb{L}(\mathbb{B}))}\Phi^T(\mathbb{C}) \tag{B2′}$$

(ii′) If $\mathbb{L}\subset\mathbb{K}$ is arbitrary, the limit $\lim_{\tilde{\mathbb{L}}\in\mathscr{K}^f,\tilde{\mathbb{L}}\nearrow\mathbb{L}}\rho_{\tilde{\mathbb{L}}}(\partial)=\rho_{\mathbb{L}}(\partial)$ exists and again

$$\rho_{\mathbb{L}}(\partial)=\sum_{\mathbb{C}\subset\mathbb{L}}\Delta_\partial(\mathbb{C}) \tag{B5′}$$

Proof of Theorem. Part (i) is proven in Ref. 16. We can also follow the proof here by proving that Δ_∂ defined by (B6) satisfies (B5), that it is unique, and that it satisfies the factorization property. To prove the estimate (B7) we use (B6), the definition of $\rho_{\mathbb{L}}(\partial)$, and (B2) to show (for $\partial\subset\mathbb{C}$)

$$\begin{aligned}\Delta_\partial(\mathbb{C})&=\sum_{\substack{\mathbb{B}\subset\mathbb{C}\\ \mathbb{B}\supset\partial}}(-1)^{|\mathbb{C}|-|\mathbb{B}|}\,\Phi(\partial)\exp\Big[-\sum_{\substack{\mathbb{D}\subset\mathbb{B}\\ \mathbb{D}\,\imath\,\partial}}\Phi^T(\mathbb{D})\Big]\\
&=\Phi(\partial)\sum_{\substack{\mathbb{B}\subset\mathbb{C}\\ \mathbb{B}\supset\partial}}(-1)^{|\mathbb{C}|-|\mathbb{B}|}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{\mathbb{D}_1,\dots,\mathbb{D}_n\subset\mathbb{B}\\ \mathbb{D}_1\imath\partial,\dots,\mathbb{D}_n\imath\partial}}\prod_{l=1}^{n}[-\Phi^T(\mathbb{D}_l)]\\
&=\Phi(\partial)\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathbb{B}:\,\partial\subset\mathbb{B}\subset\mathbb{C}}(-1)^{|\mathbb{C}|-|\mathbb{B}|}\sum_{\substack{\mathbb{D}_1,\dots,\mathbb{D}_n\subset\mathbb{B}\\ \mathbb{D}_1\imath\partial,\dots,\mathbb{D}_n\imath\partial}}\prod_{l=1}^{n}[-\Phi^T(\mathbb{D}_l)]\\
&=\Phi(\partial)\sum_{n=0}^{\infty}\sum_{\substack{\mathbb{D}_1,\dots,\mathbb{D}_n\subset\mathbb{C}\\ \mathbb{D}_1\imath\partial,\dots,\mathbb{D}_n\imath\partial\\ (\bigcup\mathbb{D}_l)\cup\partial=\mathbb{C}}}\prod_{l=1}^{n}[-\Phi^T(\mathbb{D}_l)]\end{aligned}$$

In the last equality we used the fact that

$$\sum_{\mathbb{B}:\,(\bigcup\mathbb{D}_l)\cup\partial\subset\mathbb{B}\subset\mathbb{C}}(-1)^{|\mathbb{C}|-|\mathbb{B}|}=(-1)^{|\mathbb{C}\setminus(\bigcup\mathbb{D}_l)\cup\partial|}=0$$

whenever $(\bigcup \mathbb{D}_l) \cup \partial \subsetneqq \mathbb{C}$. Thus,

$$\begin{aligned}
&\sum |\Delta_\partial(\mathbb{C})| \, e^{\omega l(\mathbb{C})} \\
&\quad \leqslant |\Phi(\partial)| \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbb{C} \in \mathscr{K}^f} \sum_{\substack{\mathbb{D}_1,\dots,\mathbb{D}_n \subset \mathbb{C} \\ \mathbb{D}_1 \wr \partial,\dots,\mathbb{D}_n \wr \partial \\ (\bigcup \mathbb{D}_l) \cup \partial = \mathbb{C}}} \prod_{l=1}^{n} [|\Phi^T(\mathbb{D}_l)| \, e^{\omega l(\mathbb{D}_l)}] \, e^{\omega l(\partial)} \\
&\leqslant |\Phi(\partial)| \, e^{\omega l(\partial)} \exp\left[\sum_{\mathbb{D} \wr \partial} |\Phi^T(\mathbb{D})| \, e^{\omega l(\mathbb{D})} \right] \\
&\leqslant |\Phi(\partial)| \, e^{\omega l(\partial) + a(\partial)}
\end{aligned}$$

according to (B4).

A particular situation where one can apply the above theorem concerns contour models used in the Pirogov–Sinai theory. Contours γ are then connected subsets $\operatorname{supp} \gamma$ of a lattice $\mathbb{Z}^\nu$ together with a configuration on it (see Definition 2.1), and one considers for $\mathbb{K}$ the set $\mathbb{K}_q$ of all q-contours with a fixed boundary condition x^q. In this case we also use the notation $\mathscr{K}_q^{\text{co}}$, $\mathscr{K}_q^f$, $\mathscr{K}_q^{\text{cl}}$, etc. Two contours γ_1, γ_2 are incompatible if $d(\operatorname{supp} \gamma_1, \operatorname{supp} \gamma_2) \leqslant 1$. If $\partial \in \mathscr{K}_q^{\text{co}}$, the set $\theta(\partial)$ of external contours of ∂ is the set $\theta(\partial) = \{\gamma \in \partial \mid \bar{\gamma} \in \partial,\ \bar{\gamma} \neq \gamma$ implies $\operatorname{supp} \gamma \subset \operatorname{Ext} \bar{\gamma}\}$. If $V \subset \mathbb{Z}^\nu$, let us introduce $\mathbb{K}_q(V) = \{\gamma \in \mathbb{K}_q \mid \operatorname{supp} \gamma \subset\subset V\}$. Whenever $V \subset \mathbb{Z}^\nu$ is finite, we denote $\mathscr{Z}_q(V) = \mathscr{Z}(\mathbb{K}_q(V))$, $\rho_{q,V}(\partial) = \rho_{\mathbb{K}_q(V)}(\partial)$ and by $\mathscr{Z}_q(V \mid \Phi, b)$ the partition function of the contour model Φ with parameter $b \geqslant 0$ defined as

$$\mathscr{Z}_q(V \mid \Phi, b) = \sum_{\partial \in \mathscr{K}_q^{\text{co}}(V)} \exp\left(b \left| \bigcup_{\gamma \in \theta(\partial)} \operatorname{Int} \gamma \right| \right) \phi(\partial) \tag{B9}$$

If $V \subset \mathbb{Z}^\nu$ (not necessarily finite), we denote by $\mathscr{K}_q^{\text{co}}(V)$ the set $\{\partial \in \mathscr{K}_q^{\text{co}} \mid \partial \subset \mathbb{K}_q(V)\}$ and by $\mathscr{K}_q^{\text{co}}(\partial, V)$ the set $\{\bar{\partial} \in \mathscr{K}_q^{\text{co}}(V) \mid \bar{\partial} \supset \partial\}$ whenever $\partial \in \mathscr{K}_q^{\text{co}}(V)$. Let us note that the set $\mathscr{K}_q^{\text{co}}(V)$ may be considered as a measurable subspace of a compact metric space $\{0, 1\}^{\mathbb{K}_q}$ endowed with its Borel σ-algebra. Finally, by $\mathscr{K}_q^{\text{a}}(V)$ we denote the set of admissible $\partial \in \mathscr{K}_q^{\text{co}}(V)$, i.e., those $\partial \in \mathscr{K}_q^{\text{co}}(V)$ for which either $\theta(\partial)$ is nonempty or ∂ is empty. Notice that $\theta(\partial) \neq \varnothing$ implies that any element of ∂ is external or is contained in the interior of some external contour.

Let us denote by $|\gamma|$ the number of lattice sites in $\operatorname{supp} \gamma$. It is easy to show that there exists a constant c such that

$$|\{\gamma \in \mathbb{K}_q \mid \operatorname{supp} \gamma \ni i,\ |\gamma| = n\}| \leqslant c^n \tag{B10}$$

for every $i \in \mathbb{Z}^\nu$. It is clear that we can put $c = |S| \, c(\nu)$, where $c(\nu)^n$ is an upper bound on the number of connected subsets of cardinality n that contain a fixed site $\mathbb{Z}^\nu$. It follows easily from the existence of a path that covers a connected graph and utilizes each its edge at most twice[20] that one may take $c(\nu)=(3^\nu-1)^{2(3^\nu-1)}$. Taking $a(\gamma)=l(\gamma)=|\gamma|$ [this is not an optimal choice; considering $a(\gamma)=a\cdot|\gamma|$ and optimizing in a, the estimates could be improved] and noticing that supports of "shortest" contour contain $(2R+1)^\nu$ lattice sites, we have the following:

Theorem B.2. Let $|\Phi(\gamma)|\leqslant e^{-\tau|\tau|}$ for each $\gamma\in\mathbb{K}_q$ with $\tau\geqslant 1+\log(2c)$. Then:

(i), (ii) The statements (i), (ii) of Theorem B.1 are fulfilled with the estimates (B4) and (B7) replaced by

$$\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \mathrm{supp}\,\mathbb{C}\ni i}} |\Phi^T(\mathbb{C}|\, e^{\omega\|\mathbb{C}\|}\leqslant 1 \qquad (i\in\mathbb{Z}^\nu) \tag{B4'}$$

whenever

$$\omega\leqslant\tau-\left[1+\log(2c)+\frac{\nu\log(2R+1)}{(2R+1)^\nu}\right]$$

(here $\mathrm{supp}\,\mathbb{C}=\bigcup_{\gamma\in\mathbb{C}}\mathrm{supp}\,\gamma$ and $\|\mathbb{C}\|=\sum_{\gamma\in\mathbb{C}}|\gamma|$), and by

$$\sum|\Delta_\partial(\mathbb{C})|\, e^{\omega\|\mathbb{C}\|}\leqslant e^{(\omega+1)\|\partial\|}\,\Phi(\partial) \tag{B7'}$$

whenever $\omega\leqslant\tau-[1+\log(2c)]$.

(iii) Whenever $V\subset\mathbb{Z}^\nu$ (in particular $V=\mathbb{Z}^\nu$) and $\partial\in\mathscr{K}_q^{f,\mathrm{co}}(V)$, the limit over finite $U\subset V$, ordered by inclusion, of $\rho_{q,U}(\partial)$ exists

$$\lim_{U\nearrow V}\rho_{q,U}(\partial)=\rho_{q,V}(\partial)$$

If $\Phi(\gamma)\geqslant 0$ for every $\gamma\in\mathbb{K}_q$, there exists a unique σ-additive probability measure $P_{q,V}$ on $\mathscr{K}_q^{\mathrm{co}}(V)$ such that

$$P_{q,V}(\mathscr{K}_q^{\mathrm{co}}(\partial,V))=\rho_{q,V}(\partial)$$

for each $\partial\in\mathscr{K}_q^{\mathrm{co}}(V)$. Moreover, $P_{q,V}(\mathscr{K}_q^{\mathrm{a}}(V))=1$ and $P_{q,V}$ is the weak limit

$$P_{q,V}=\lim_{U\nearrow V}P_{q,U}$$

(iv) Assuming furter that Φ is translation-invariant, one has for each finite $V\subset\mathbb{Z}^\nu$

$$\log\mathscr{Z}(V)=p(\Phi)\,|V|-\sum_{\substack{\mathbb{C}\in\mathscr{K}_q^{\mathrm{cl}}\\ \mathrm{supp}\,\mathbb{C}\cap V^c\neq\varnothing}}\Phi^T(\mathbb{C})\,\frac{|\mathrm{supp}\,\mathbb{C}\cap V|}{|\mathrm{supp}\,\mathbb{C}|}$$

with

$$p(\Phi) = \sum_{\substack{\mathbb{C} \in \mathscr{K}_q^{\mathrm{cl}} \\ \mathrm{supp}\, \mathbb{C} \ni i}} \frac{\Phi^T(\mathbb{C})}{|\mathrm{supp}\, \mathbb{C}|} \tag{B11}$$

and

$$|\log \mathscr{Z}(V) - p(\Phi)|V|| \leqslant \{\exp[-\omega(2R+1)^\nu]\}\ |\partial V| \tag{B12}$$

whenever

$$\omega \leqslant \tau - \left[1 + \log(2c) + \frac{\nu \log(2R+1)}{(2R+1)^\nu}\right]$$

Proof (cf. Ref. 15). It is straightforward to verify (B1) if $\tau \geqslant 1 + \log(2c)$. Then (i) and (ii) follow directly from Theorem B.1.

The existence of $\lim_{U \nearrow V} \rho_{q,U}(\partial)$ in (iii) is implied with the help of (B5) and (B7). [This assertion is actually contained in (ii′).] To introduce a contour probability measure on $\mathscr{K}_q^{\mathrm{co}}(V)$, we denote by $\mathscr{M}_\partial$ the (closed) Borel set $\mathscr{M}_\partial = \mathscr{K}_q^{\mathrm{co}}(\partial, V)$ and introduce the probability $P_{q,U}$ on $\mathscr{K}_q^{\mathrm{co}}(V)$ [supported by $\mathscr{K}_q^{\mathrm{co}}(U)$] by

$$P_{q,U}(\partial) = \chi_{\mathscr{K}_q(U)} \frac{\Phi(\partial)}{\mathscr{Z}(U)}$$

Any cylinder set is a finite, disjoint union of sets of the form $\mathscr{M}_{\mathbb{M},\partial} = \{\tilde{\partial} \in \mathscr{K}_q(V) \mid \tilde{\partial} \cap \mathbb{M} \subset \partial\}$ with $\mathbb{M} \in \mathscr{K}_q^f(V)$ and $\partial \in \mathscr{K}_q^{\mathrm{co}}(\mathbb{M})$. Since

$$\mathscr{M}_{\mathbb{M},\partial} = \mathscr{M}_\partial \Big\backslash \bigcup_{\gamma \in \mathbb{M} \backslash \partial} \mathscr{M}_{\mathbb{M},\gamma}$$

and

$$\bigcap_{\gamma \in \tilde{\partial}} \mathscr{M}_{\partial \cup \{\gamma\}} = \mathscr{M}_{\partial \cup \tilde{\partial}}$$

for $\partial \cup \tilde{\partial}$ compatible and $\mathscr{M}_{\partial \cup \tilde{\partial}} = \varnothing$ otherwise, one has

$$\begin{aligned} P(\mathscr{M}_{\mathbb{M},\partial}) &= P(\mathscr{M}_\partial) - P\left(\bigcup_\gamma \mathscr{M}_{\partial \cup \{\gamma\}}\right) \\ &= P(\mathscr{M}_\partial) - \sum_{\substack{\tilde{\partial} \in \mathbb{M} \backslash \partial \\ \tilde{\partial} \neq \varnothing}} (-1)^{|\tilde{\partial}|+1} P(\mathscr{M}_{\partial \cup \tilde{\partial}}) \\ &= \sum_{\tilde{\partial} \subset \mathbb{M} \backslash \partial} (-1)^{|\tilde{\partial}|} P(\mathscr{M}_{\partial \cup \tilde{\partial}}) \end{aligned}$$

for any probability P on $\mathscr{K}_q^{\mathrm{co}}(V)$.

Thus, it suffices to verify the convergence $P_{q,U} \to P_{q,V}$ only on $\mathcal{M}_\partial$: $P_{q,U}(\mathcal{M}_\partial) = \rho_{q,U}(\partial) \to \rho_{q,V}(\partial) = P_{q,V}(\mathcal{M}_\partial)$.

To verify that $P_{q,V}(\mathcal{K}_q^{a}(V)) = 1$, we refer to the usual proof (cf. Proposition 2.2 in Ref. 2). Indeed, note first that considering a half-line parallel to a fixed coordinate axis of $\mathbb{Z}^\nu$ and starting at a fixed site $i \in \mathbb{Z}^\nu$, there are fewer than n possibilities for its first intersection with a contour γ encircling the site i such that $|\gamma| = n$ and thus

$$\begin{aligned} &P_{q,V}(\{\partial \mid i \in \operatorname{Int} \gamma, |\gamma| = n, \gamma \in \partial\}) \\ &\quad \leqslant n P_{q,V}(\{\partial \mid i \in \operatorname{supp} \gamma, |\gamma| = n, \gamma \in \partial\}) \end{aligned} \tag{B13}$$

(cf. proof of Lemma 2.7 in Ref. 2).

Hence, the probability that a site i is encircled by at least n contours may be bounded by

$$\begin{aligned} &\sum_{m=n}^{\infty} P_{q,V}(\{\partial \mid i \in \operatorname{Int} \gamma, |\gamma| = m, \gamma \in \partial\}) \\ &\quad \leqslant \sum_{m=n}^{\infty} m c^m \exp(-\tau m) \\ &\quad \leqslant \sum_{m=n}^{\infty} 2^m c^m \exp(-\tau m) = \frac{\exp\{-[\tau - \log(2c)] n\}}{1 - \exp\{-[\tau - \log(2c)]\}} \end{aligned}$$

since the length of the nth contour encircling i is at least n and $m < 2^m$. Then the probability that the site i is encircled by an infinite number of contours is bounded by

$$\lim_{n \to \infty} \frac{\exp\{-[\tau - \log(2c)] n\}}{1 - \exp\{-[\tau - \log 2c)]\}} = 0$$

Finally, the statement (iv) is proved by a direct application of (B2) and (B4′).

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REFERENCES

1. S. A. Pirogov and Ya. G. Sinai, *Teor. Mat. Fiz.* **25**:358–369 (1975); **26**:61–76 (1976) [*Theor. Math. Phys.* **25**:1185 (1975); **26**:39 (1976)].
2. Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results* (Akademiai Kiado, Budapest, 1982).

3. R. L. Dobrushin, *Teor. Veroyat. Primen.* **17**:619–639 (1972) [*Theory Prob. Appl.* **17**:582 (1972)].
4. (a) J. Bricmont, J. L. Lebowitz, C. E. Pfister, and E. Olivieri, *Commun. Math. Phys.* **66**:1–20 (1979); (b) J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *Commun. Math. Phys.* **66**:21–36; **69**:267–291 (1979).
5. C. Borgs, *Commun. Math. Phys.* **96**:251–284 (1984).
6. J. Bricmont, K. Kuroda, and J. L. Lebowitz, *J. Stat. Phys.* **33**:59–75 (1983).
7. J. Bricmont and J. Fröhlich, *Commun. Math. Phys.* **98**:553–578 (1985).
8. M. Zahradník, in *Proceedings of Les Houches 1984 Summer School. Critical Phenomena, Random Systems and Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, to appear).
9. R. Kotecký and D. Preiss, *Commun. Math. Phys.* **103**:491–498 (1986).
10. R. L. Dobrushin, *Teor. Veroyat. Primen.* **13**:201–229 (1968) [*Theory Prob. Appl.* **13**:197 (1968)]; J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *J. Stat. Phys.* **21**:573–581 (1979).
11. H. van Beijeren, *Commun. Math. Phys.* **40**:1 (1975).
12. R. L. Dobrushin, *Teor. Veroyat. Primen.* **18**:261–279 (1973) [*Theory Prob. Appl.* **18**:253 (1973)].
13. R. Kotecký and D. Preiss, *Rend. Circ. Mat. Palermo, Ser. II* **3**:161–164 (1984).
14. D. Preiss, unpublished; M. Zahradník, *Commun. Math. Phys.* **93**:559–581 (1984).
15. R. Kotecký and D. Preiss, unpublished.
16. J. Navrátil, Contour models and unicity of random fields, Thesis, Charles University, Prague (1982) [in Czech].
17. K. Kuratowski, *Topology*, Vol. 2 (Academic Press, New York, 1968).
18. C. Gruber and H. Kunz, *Commun. Math. Phys.* **22**:133–161 (1971).
19. G. Gallavotti, A. Martin-Löf, and S. Miracle-Solé, in *Battelle Seattle 1971 Rencontres*, A. Lenard, ed. (Lecture Notes in Physics, Vol. 20, Springer, Berlin, 1973), pp. 162–204.
20. O. Ore, *Theory of Graphs* (AMS Colloquium Publication, Vol. 38, Providence, Rhode Island, 1962).